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Control Charts

AN INTRODUCTION TO STATISTICAL QUALITY CONTROL

by

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CONTROL CHARTS

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PREFACE

This text is intended for men and women who wish to acquire knowledge of the methods and charts of statistical quality control, but who are too busy to have an interest in the mathematical aspects of the methods. Consequently, emphasis has been placed upon practical applications rather than upon the development of mathematical formulas. It has been assumed that the reader will accept without proof such mathematical formulas as are used. For this reason mathematical analyses and computations have been kept at a minimum.

In Chap. I, which is an introductory chapter, stress is placed upon the importance of statistical methods as an aid in the control of quality. During the Second World War the methods of statistics proved invaluable as a means of increasing the production of high-caliber goods. The applications of the methods were in most cases also accompanied by an appreciable decrease in cost, though this was considered less important. Now that the war has ended, production of goods of high quality at low cost continues to be of vital concern to the manufacturer, faced as he is with the strenuous conditions of peacetime competition. Hence, it is more important than ever that everyone concerned with such production should be able to make and read control charts.

In Chap. II frequency curves are studied, and their use in analyzing statistical data is considered. It is here explained that limits for the measurements of manufactured items may be revealed by use of these curves.

In Chaps. III and IV all the charts except one that are used with the control of quality by statistical methods are presented. Some of the formulas given in these chapters differ slightly from those of customary practice. However, the results obtained are the same as those found by the use of standard formulas. The changes in formulas or presentation have been made to avoid, as far as possible, the use of mathematical formulas and methods that might confuse the reader no longer familiar with the operations and practices of elementary algebra.

In Chap. V all formulas that were used in the earlier chapters are identified with corresponding ones of usual practice, and the chart not given in Chap. III is presented. In order to give the standard formulas in Chap. V it is necessary to extend the use of mathematical considerations beyond the minimum use of the first four chapters.

It will be observed that in the body of the text the figure numbers and table numbers correspond to the numbers of the articles in which they occur. Three general tables that are used frequently throughout the text are given after Chap. V and are numbered Table I, Table II, and Table III.

Table II is a new table that has been developed to save labor in calculating the value of $3\sqrt{[p(1 - p)]/n}$. This radical expression is needed in determining the location of control limits for charts for fraction defectives, and Table II furnishes a simple means for finding its value for various values of p and n.

The author has endeavored to present the material on control charts that was included in the standard intensive courses that were conducted during the Second World War through the Office of Production, Research, and Development of the War Production Board and under the direction of Dr. Holbrook Working, of Stanford University, and Dr. Edwin G. Olds, of the Carnegie Institute of Technology; he wishes to express his appreciation for the help he received from these two prominent mathematicians, from Dr. Irving W. Burr, of Purdue University, from Dr. Cecil C. Craig, of the University of Michigan, and from Paul Beach, Head, Industrial Sections, Institute of Statistics, University of North Carolina, Raleigh, while attending an intensive course at Purdue University in 1944. He also wishes to express his appreciation for their advice concerning practical applications and the design of forms for tables and charts to Arthur Bender, Delco-Remy Division, General Motors Corporation, Anderson, Indiana, R. J. Smart. Wright Aeronautical Corporation, Paterson, New Jersey, and Louis R. Zimov, Crosley Corporation, Cincinnati, Ohio. The author is particularly indebted to his colleague, Dr.

Harry L. Miller, professor of mathematics, University of Cincinnati, who read the manuscript and made many valuable suggestions, thereby improving both the material and its presentation.

EDWARD S. SMITH

CINCINNATI, OHIO,
May, 1947.

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CONTROL CHARTS

CHAPTER I

STATISTICAL QUALITY CONTROL

1. Introductory Remarks

Quality control by statistical methods is now so extensively applied in all lines of industry, and in all sections of the United States, that everyone who is interested in manufacturing should also have a definite interest in the methods. The methods, and the control charts based on them, were developed [1] over twenty years ago by Dr. W. A. Shewhart in the Bell Telephone Laboratories. They were applied in the Western Electric plant in 1924 and there was an immediate improvement in the condition of scrap; the reduction in rejects amounted to as much as 50 per cent on some items. There was also a resulting saving in millions of dollars in cost of operation.

Other large corporations, with well-manned research departments, adopted the methods and each received extensive benefits through an increase in production, an improvement in quality, and a reduction in costs. When other manufacturers learned of the economic advantages to these corporations that resulted from the use of these methods, many became interested and introduced statistical quality control in their own plants. However, it required the tremendous increase in production of the Second World War to cause the methods to become widely used.

2. Training Program as a War Measure

Increase in production to satisfy the Second World War's demands, mass production with new and untried personnel, more stringent requirements for accuracy in meeting exacting specifications set by the Army and the Navy, and the necessity for interchangeability of parts that were made by different manufacturers, all made the control of quality more important than it had been before the Second World War began. As a consequence, and in order to give all manufacturers who were engaged in war production an opportunity to benefit by using statistical methods in controlling the quality of their products, an intensive program of training was presented under the War Production Board. Eight-day courses were given in all sections of the country, and the methods taught were soon being successfully applied in hundreds of factories. Usually in each factory where the methods were introduced there was an improvement in quality with an accompanying increase in the production of acceptable items.

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[1] See "Quality Control" in *Fortune*, October, 1943.

3. Statistical Quality Control and Peacetime Production

During the Second World War it was imperative not to allow the cost of production to interfere with quantity produced or the resulting quality of the product. Now that the Second World War has been won the manufacturer will have few contracts for the government and will not have to meet Army and Navy specifications with everything that he makes, but he will have to produce goods of high quality under strenuous peacetime competitive conditions. To compete with other manufacturers he must be able to maintain high standards of quality while at the same time to reduce his costs to a minimum.

Quality control by statistical methods was called upon during the Second World War to aid in increasing the production of goods of high quality. It was generally found that an application of the methods not only resulted in increasing production and in improving quality, but also in an appreciable decrease in cost. All of these factors, production, quality, and cost, will be of as vital concern to the manufacturer under peacetime conditions as they were during the Second World War.

4. Cooperation

For a really successful application of statistical methods in the control of quality it is essential that the quality control department have the full cooperation of all other departments. There are, of course, three important groups that are vitally concerned with the success of a control program: (*a*) The executive officers, (*b*) those in the quality control department, and (*c*) those in the engineering and the production departments who are responsible for making the items the quality of which is to be controlled. We shall refer to these as "management," "quality control engineers," and "production," respectively.

The quality control engineers who are in charge of the program should have the full training necessary for them to qualify as experts.

The men in management and in production cannot take the time that would be required to gain the full detailed familiarity with statistical quality control that the quality control engineers must have. Nor is this necessary. However, the executives should have sufficient knowledge to allay any skepticism that they may have of the methods, and the operators should gain enough knowledge to prevent them from being suspicious or fearful of any effect that using the methods might have upon them or their working conditions. When this condition prevails then satisfactory cooperation between all departments concerned will be assured.

5. A Text for the Busy Executive

The busy executive must make many quick decisions every day on matters affecting plant policy and production procedure, and to do this he must have a wide knowledge of all phases of factory operation. He cannot accept a process just because someone says it will work, but he must be convinced of its worth. Hence, if he is to approve a quality control program based on statistical methods, he must be satisfied that through use of the methods the production in his plant will be improved and that this improvement will cause no increase in cost. There is no more effective way for him to become convinced that production will be improved and costs actually will be reduced than for him to learn how to make and to read the control charts that result from an application of statistical quality control. But he cannot afford the time that would be required to refresh his mind on a mass of mathematical theory with which he has long since become unfamiliar.

In the present text an attempt will be made to give a brief, but complete, presentation of the charts of statistical quality control, and to have that presentation as free of mathematical theory and formulas as possible. All statements and formulas that are based on mathematical theory have been proven true in courses on statistical methods, and should be accepted here without proof by the man who is interested only in the applications of the methods.

While this text is intended for the busy executive, the presentation has been modified sufficiently to make it suitable for in-plant training of the operator who should know something about the charts, but who may have had a limited training in mathematics. A knowledge of the charts would enable this practical man to supplement his common sense, judgment, and experience and to attain the highest degree of quality in his work. He will then be able to cooperate better with the quality control engineers in putting on a full control program.

6. Definitions

Since the terms "control," "quality," and "satisfactory control of quality" will be used extensively, we shall now define them. The definitions will be limited to the sense in which the terms defined will be used in this text. The amplifying statements will be as brief as consistent with necessary clarification. The definitions are intended to apply to a product that is manufactured under controlled conditions and to continue to apply as long as the conditions of manufacture remain unchanged.

a. Control.—*Let us define a controlled manufacturing process as one in which the variations in the values of the items from a central, or average, value are kept as small as possible and remain within expressible limits.*

The average value referred to in the definition is the actual arithmetic average obtained from the manufactured items and is seldom exactly equal to the average as given in the specifications.

The definition applies whether we are controlling a dimension of a machined part, the analysis of a chemical compound, the tensile strength of a metal, the smoothness of a surface, or any other specified property or characteristic.

Let us take, for example, an item for which a dimension is to be controlled. Say we are manufacturing bushings whose outside diameters are to lie within the range from 0.997 inch to 1.003 inches; an equivalent expression for this range is 1 inch ± 0.003 inch.

It cannot be said that the process is under control merely because the bushings inspected pass a go, no-go-gage. That would, of course, determine those bushings to be accepted and those to be rejected under specifications, but even if all inspected were acceptable it would give no assurance that the process would continue to make acceptable bushings. To gain this assurance it is necessary to know the distribution of the various sizes.

Even with the greatest care possible no two bushings can be made exactly alike. If measurements are carried to a sufficient number of decimal places, it will be found that no two bushings will have the same diameter. However, if the diameters are measured only to the nearest thousandth of an inch (or some other small fraction of a unit) then it will be found that there are many having the same diameters. In thousandths of an inch these diameters will vary from 0.997 inch to 1.003 inches if the bushings are meeting specification requirements. If the process is under satisfactory control then there would be more bushings whose diameters would be recorded as 1 inch than there would of any other size, there would be more at 0.999 inch and 1.001 inches than at 0.998 inch or 1.002 inches, more at 0.998 inch or 1.002 inches than at 0.997 inch or 1.003 inches, and rarely, or never, any smaller than 0.997 inch or larger than 1.003 inches.

For the bushings whose diameters we are considering, the variations in the size of the diameters from the average size are variables through whose behavior the condition of control of the process can be determined. If the process is under control then these variations of the diameters from the average will obey the general laws of distribution that govern the behavior of variables, which are subject to laws of probability. These variations in the diameters obey the same laws that are followed by the variations in the number of coins that come heads up if 10 coins are tossed repeatedly.

We shall see in Chap. III that to determine whether or not a process is under control it is necessary to obtain actual measurements of enough items (taken from the production line in order of production) to establish the distribution of the measurements of all the items. We shall also see how to fix the limits (indicated in the definition of "control") within which the items must remain.

With the determination and use of these limits we shall be using statistical methods in the control of quality.

b. Quality.—*It is difficult to give a concise definition of quality, for any definition that might be given would be inadequate* and would require interpretations. Nevertheless, we shall give two very limited definitions here. These will express what we shall consider as satisfactory quality: one definition for a single item, the other for a production process.

The quality of a single item, with regard to one of its characteristics, is satisfactory if that characteristic meets specification requirements.

A production process is said to be producing goods of satisfactory quality if 99.7 per cent of the items are satisfactory in quality.

c. Satisfactory Control of Quality.—*Let us define a condition of satisfactory control of quality of a manufacturing process as one in which the process is under control and the quality of the product is satisfactory.*

Under the definition of "control," as given above, the average and the limits are set by the process and not by the specifications. If the average thus set is approximately that set by specifications and the expressible limits for variations from the average are satisfactorily within specification limits, then the control of quality for that process is satisfactory.

For the example of the bushings considered above, the average diameter as set by the process should differ only slightly from the 1 inch as set by the specifications, and the corresponding limits for variations from this average should be within specification limits. Suppose that the average outside diameter as set by the process is 1.0004 inches and that the corresponding limits for variations from this average are 0.9982 inch and 1.0026 inches (obtained by methods to be explained in later chapters), then, since the corresponding specification values are 1.0000 inch, 0.9970 inch, and 1.0030 inches, the process would be under satisfactory control.

d. Comment.—The definitions in *a*, *b*, and *c*, and the discussion that followed each, have been as fully expressed as is possible at this time. The general presentation of control limits and the methods of computing them, which will appear in subsequent chapters, will add much to our understanding of the terms used here. It will be found that a process can be under control and the control not be satisfactory. The control can be unsatisfactory because *the average is too large (or too small)*, or because the spread between the maximum and the minimum values of the characteristic being measured is too great, or both.

Comment should be made here regarding the slang expression "Everything is under control" which is commonly used in the United States to imply that everything is quite satisfactory. Perhaps it is unfortunate that in statistical quality control it is customary to say that a process is under control if the measurements remain within determinable limits, regardless of whether or not the limits are satisfactory. If the reader is careful to note that the process must be under control, and that the control must be satisfactory, then he should have no confusion over the use of the term "under control."

7. Chart Control vs. Gage Control

The reader may say that for many years, mass production and the necessity for interchangeability of parts have made manufacturers strive to attain a condition of satisfactory control of the quality of their product. The answer is, of course, that that is true. As a result a high grade quality control department has become an important department in every modern manufacturing plant, and applications of statistical methods constitute only one phase of the work of such a department. However, until statistical methods were applied to the control of quality, the control was entirely by inspection. This was a negative control as it told only how good or how bad the item was after it had been made.

By carefully inspecting each item it is possible to detect and discard those items that do not meet specification requirements (although even with 100 per cent inspection some items of inferior quality always slip through and are accepted), but you cannot tell from this inspection what condition of quality to expect in items yet to be made. In this connection a statement that often appears in writings on quality control is repeated here: "Quality cannot be inspected into an item but must be put there in the process of manufacturing it."

By the use of charts based on statistical methods it is possible (by inspecting samples) to establish whether or not the process is under control, and if it is not, to discover when and where to look for trouble. If the process is under satisfactory control, or if causes for trouble have been found, corrected, and the process has been brought under satisfactory control, then the manufacturer can be assured that, so long as there is no change in the production process, the items as made will be satisfactory in quality.

Control by inspection may be classified as *gage control,* while control by using statistical methods is *chart control.*

The chief difference between gage control and chart control is that gage control is concerned with the finished product, while chart control is concerned with the process.

Gage control attempts to detect machine failures after they occur, and then to separate the good from the bad.

Chart control deals with future production, and always endeavors to predict machine failures, or other process failures, before they occur, and to prevent, as far as possible, defective items from being produced.

Gage control asks, "How bad can a product be and still be acceptable?"

Chart control asks, "How good can a process become before it is economically unsound to try and improve it?"

Gage limits are set by the specifications.

Control limits on a chart are set by the process.

It will be shown in Chap. III that when the process is under satisfactory control the control limits must be an appropriate distance inside of specification limits.

8. Charts

Charts of various kinds are so widely used that everyone should be familiar with the important features of the more common ones. The control charts, which will be introduced in Chap. III, will be found to resemble a patient's temperature chart very closely. Just as a patient's temperature chart will tell the doctor an important continuing story regarding the patient's condition, so a control chart will tell those who are familiar with control charts a continuing story of the state of quality of the manufactured items.

A doctor, with a patient whose sickness extends over several weeks or months, will frequently wish to review the temperature history of the case. From the temperature chart, which is always kept by the nurses, the doctor can see at a glance significant readings of temperature, temperature ranges, periods of the day when certain temperature changes occurred, etc., and can gain much valuable information that would require a great deal more time if he had to look at tabulated figures. More important than the time the examination would take is the matter of accuracy. The doctor cannot overlook important features on the chart, while his glance could slide past a significant entry in a tabulation. With a chart it is very easy to make a comparison of temperatures and temperature variations for different periods over several weeks, while it would be difficult, if not impossible, to make the same comparison by examining the tabulated figures.

A control chart is very much like this temperature chart, and serves the same general purposes. On a control chart there is usually shown an average, or a representative value, that compares with the normal line on the temperature chart. Points that represent the average measurement of a sample of a small number of manufactured items are plotted on the chart, and all these points should be located within certain control limits. These limits are determined by methods developed in Chap. III. If a point falls outside of a control limit then a search should be made to find what has gone wrong with the process.

An individual usually doctors himself when he has minor aches or pains, taking his favorite headache cure, a cathartic, or some other remedy, or he does nothing, according to the individual and the way he feels. If, while he had a slight headache or some other mild pain, his temperature chart were plotted the points would probably fall above normal, but he rarely checks his temperature in these cases because he does not feel sick enough to call a doctor, and he thinks he can take care of himself. As often as not, the medicine that he takes has no effect upon his cure, for nature looks after the human machine and heals minor ills. Sometimes the medicine taken is quite harmful and eventually a doctor may be needed to overcome the damage done. If the individual's temperature should go very far above normal, he would know that he is a sick man, even without reading his temperature. He would hasten to call a reputable physician, preferably one who has gained his confidence through years of competent service. He would rely upon the doctor's skill as he forms his diagnosis and as he prescribes for the cure of the disease.

A control chart will tell a story for the manufacturing process for which it is made, similar to the story that would be told for an individual's physical condition by his temperature chart, if that chart were kept at all times. Unfortunately nature cannot look out for a manufacturing process, and a control chart should be kept for those items for which it is important to maintain a satisfactory condition of control, and for which it may be difficult to produce items of satisfactory quality. If points remain within control limits, and do not indicate a trend that should warn of approaching trouble, then we should do as the individual does, when he feels all right or just slightly indisposed, that is, do nothing. But if a dangerous trend is seen, or if points fall outside of control limits, then the operator should know that the process is sick, and he should immediately look for trouble. If the trouble is too serious for the operator or his supervisor to find and fix then call in the doctor--call in the quality control engineer.

This illustrates two important principles of a quality control analysis:

1. *Take no action* if the control chart indicates that no action is needed.

2. *Take action* if the control chart indicates the presence of trouble, or pending trouble.

Everyone in the plant, from the executives in the front office to the operators who share in turning out the finished product, should have a common interest in maintaining satisfactory control of quality. They should have faith in control charts and should be able to read and to interpret the story a chart may tell. An operator should know when the chart indicates trouble that will require more than his judgment and experience in making corrections, and on whom he should call for assistance. The final responsibility for maintaining and improving quality should rest with the quality control engineer, and his training and experience should be such that everyone will have confidence in his ability to perform the task.

9. Benefits

Manufacturers who are not now using statistical methods in the control of quality would undoubtedly do so if they could realize the benefits to be gained. In all lines of manufacturing, on large-scale production and on small, thousands have benefited. Any manufacturer who is not using the methods, and who makes a definite attempt to apply them correctly, should receive benefits that he now believes impossible.

For the information of those manufacturers who have not tried the methods, we list some of the advantages or benefits that have been enjoyed by those who have successfully applied them.

1. Psychological effect upon all who are affected by the control of quality.
2. Reduction in rejections.
3. Detection of trouble as soon as something goes wrong with the process.
4. Prediction of impending trouble. Anticipating trouble before rejections occur, and thus avoiding rejections and interruption.
5. Reduction in cost.
6. Provides better basis for establishing or altering specification requirements.
7. Provides basis for reducing to a minimum the variation between individual units for the process.
8. Provides a permanent record of quality.
9. Provides a basis for acceptance of a product by a purchaser.
10. Gives basis for greater confidence in inspection results, and enables inspection, production, and engineering departments to get together for discussion and solution of their common problems.

CHAPTER II

FREQUENCY DISTRIBUTIONS

10. Frequency Distribution

A frequency distribution is a tabulation of the frequency of occurrence of some variable. We shall be interested in distributions of variables of normal variations such as those associated with games of chance. When tossing coins, rolling dice, or dealing cards, we see excellent illustrations of normal chance variations, or of variables that obey the laws of probability or chance.

In the study of statistical methods, frequency distributions and frequency curves play important roles because from them we can gain much important information about the data under consideration. For example, we may determine whether there is a normal average for all the variables of the distribution, whether the variables are symmetrical about the average, whether the normal average is what it should be, and, in particular, we can determine limits within which practically all variables that belong to a given distribution must fall. These limits can be determined from a frequency distribution by formulas that are developed in courses on statistical methods. While we shall not develop these formulas, we shall use formulas that are based on them, and we can understand the use of these formulas better if we appreciate what a frequency curve represents.

This question of the use of frequency distributions in a statistical analysis is of importance to us because, in a manufacturing process, the quality of a characteristic of the product varies from item to item and, if the process is under control, this variation should be due to chance causes. Hence, the variables that represent this variation in quality have normal variations. These variables should obey the laws of probability, and they should have a symmetrical frequency distribution. In the next article, we shall use an experiment with tossing coins as a means of introducing frequency distributions.

11. Tossing Coins

A frequency distribution which the reader can check readily is one obtained by counting the number of heads in tossing a coin several times, or in tossing several coins at once, and then repeating this process. The probability of tossing a head with a single coin is $\frac{1}{2}$, or the chance of tossing a head is fifty-fifty, or even. This does not mean that if a head is tossed with one throw a tail will be tossed with the next, but that, for a large number of tosses, there will be approximately as many heads as tails. The laws of probability are said to work "in the long run."

For our purposes the best distribution will be obtained if we toss a few coins several times and if we keep a record of the number of heads that appear with each toss. Say we toss six pennies that are not too old and worn. The pennies should be thoroughly shaken up (preferably in an old flour sifter) so that there will be nothing outside of the laws of chance to make a coin come up heads. After the coins are thoroughly mixed toss them on a blanket and count and record the number of heads that appear. You should repeat the process until you have the results of several hundred tosses of the six coins, although, if the pennies are thoroughly mixed after each toss, you should have a fair distribution with any number of tosses from about fifty and over. A tally record should be kept, and for this purpose the form used with Table 11*a* will be found quite satisfactory.

The theoretical distribution for the number of heads per toss of 6 coins when the 6 coins are tossed 64[1] times is given in Table

---

[1] We have chosen 64 tosses of the 6 coins because with 64 tosses the frequency for each possible number of heads per toss can be expressed as an integer. This arises from the expansion of the binomial expression for the sum of the probabilities of throwing a head and of failing to throw a head with 1 toss of a single coin, that is, from

$$\left(\tfrac{1}{2} + \tfrac{1}{2}\right)^6 = \tfrac{1}{64} + \tfrac{6}{64} + \tfrac{15}{64} + \tfrac{20}{64} + \tfrac{15}{64} + \tfrac{6}{64} + \tfrac{1}{64}$$

The probability of tossing a head 6 times in 6 tosses of 1 coin (or 6 heads with 1 toss of 6 coins) is $\frac{1}{64}$, of tossing exactly 5 heads is $\frac{6}{64}$, 4 heads is $\frac{15}{64}$, 3 heads is $\frac{20}{64}$, 2 heads is $\frac{15}{64}$, 1 head is $\frac{6}{64}$, and finally no heads is $\frac{1}{64}$. If each probability is multiplied by 64 we find 1, 6, 15, 20, 15, 6, and 1 as the theoretical number of times we should find 6, 5, 4, 3, 2, 1, 0 heads, respectively, in 64 tosses of 6 coins.

11*a*. It should be observed that in the order of frequency we find 3 heads, then 4 heads or 2 heads, then 5 heads or 1 head, and finally 6 heads or 0 heads.

It is unnecessary to show the column of Table 11*a* headed "Tally Record" with a frequency table or a frequency distribution. This column is shown here, and will be shown with a few other distributions, in order to indicate how a frequency distribution is usually obtained by means of a tally record.

Table 11*a*.—Theoretical Distribution of Heads in Tossing 6 Coins 64 Times

No. of heads per toss X	Frequency of occurrence	
	Tally record	Frequency f
6	/	1
5	𝍸 /	6
4	𝍸 𝍸 𝍸	15
3	𝍸 𝍸 𝍸 𝍸	20
2	𝍸 𝍸	15
1	𝍸 /	6
0	/	1
	Total	64

The letters X and f introduced in the table have no particular significance here, but will have definite uses in subsequent applications. In the table, X refers to any one of the entries that appear in the column giving the number of heads per toss, and f to the corresponding frequency of the X value chosen. For example, for $X = 4$ we find $f = 15$.

The reader should obtain a frequency distribution by tossing 6 coins a number of times. If he tosses them 64 times he should not expect to get the exact distribution of Table 11*a*, but, if he thoroughly mixes the coins after each toss, his distribution should not vary much from the theoretical one given in the table.

Actual records of two experiments are shown in Tables 11*b* and 11*c*. For Table 11*b* the coins were held between cupped hands. It was thought that the coins were thoroughly mixed for each toss, but evidently they were not. Note the influence that one toss often had upon the next by observing that 4 heads and 2 heads appear with greater frequency than they should. Several times a toss of 4 heads would be followed by one of 4 heads or 2 heads, which would look suspiciously as though the coins came up the same as before or else were reversed.

Table 11*b*.—Actual Distribution of Heads in Tossing 6 Coins 64 Times

(*Poorly mixed after each toss*)

No. of heads per toss X	Frequency of occurrence	
	Tally record	Frequency f
6	/	1
5	𝍸 //	7
4	𝍸 𝍸 𝍸 ////	19
3	𝍸 𝍸	10
2	𝍸 𝍸 𝍸 𝍸 /	21
1	////	4
0	//	2
	Total	64

Because the distribution of Table 11*b* differed so much from a normal distribution, the coins were placed in a flour sifter and were mixed before each drawing of Table 11*c*.

Of course 64 is too small a number of tosses on which to establish results for such a distribution, but even so, the results given in Tables 11*b* and 11*c* are quite significant in showing the necessity of thoroughly mixing the coins before each toss.

The reader would do well to make a greater number of tosses, but he should be careful to thoroughly mix the pennies before each toss. If he makes his total number of tosses a multiple of 64, he can readily obtain the theoretical distribution, with which to compare his own, by multiplying the frequencies of Table 11*a* by the same factor. For example, if he tosses 6 coins 256 times then the frequencies of Table 11*a* would be multiplied by 4 to obtain the appropriate theoretical distribution. This would give the theoretical values of *f* as 4, 24, 60, 80, 60, 24, and 4 for number of heads *X* of 6, 5, 4, 3, 2, 1, 0, respectively.

Table 11c.—Actual Distribution of Heads in Tossing 6 Coins 64 Times

(*Coins thoroughly mixed in flour sifter*)

No. of heads per toss *X*	Frequency of occurrence	
	Tally record	Frequency *f*
6	/	1
5	𝍸 /	6
4	𝍸 𝍸 ////	14
3	𝍸 𝍸 𝍸 𝍸 ///	23
2	𝍸 𝍸 //	12
1	𝍸 /	6
0	//	2
	Total	64

12. Bar Diagram

A frequency distribution can be exhibited and studied advantageously by means of a frequency curve. As an approach to the frequency curve let us exhibit the data of Table 11*a* in a bar diagram. A bar diagram is the simplest form of a statistical graph. It can be readily made and interpreted, and is useful for making comparisons of the variations in the number of times that certain items appear

In Fig. 12*a* is shown a vertical bar diagram for the data of the theoretical distribution given in Table 11*a*.

To construct the diagram some care is needed in selecting the scale for the space between the bars and for the frequency scale which determines the lengths of the bars. The lengths of the bars are proportional to the frequencies of Table 11*a* and those frequencies can be read on the vertical scale. For example, if we wish to read the number of times exactly 2 of the 6 coins appeared with heads up in the 64 tosses, we select the bar at 2 on the horizontal scale, project from its upper end to the vertical scale, and find that in 15 of the 64 tosses exactly 2 of the 6 coins had heads up.

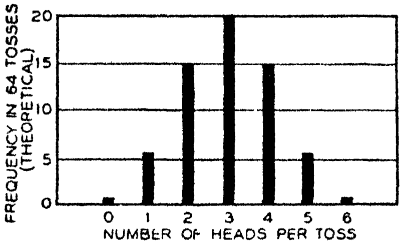

FIG. 12*a*.

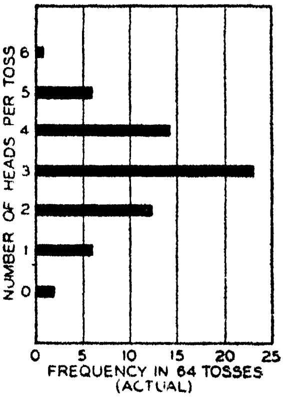

FIG. 12*b*.

Bar diagrams are frequently drawn with the bars horizontal. Such a diagram for the data of Table 11*c* is shown on Fig. 12*b*.

13. Frequency Curve

To draw a frequency curve for the data of Table 11*a* we lay off scales like those for the bar diagram of either Fig. 12*a* or 12*b*, respectively, depending upon whether we wish a curve with a horizontal or a vertical base line. For Fig. 13*a* we have chosen the form of Fig. 12*a* with a horizontal base line and have marked scales as shown. Instead of drawing a bar at each point of division on the base line we placed a dot at the position where the center of the upper end of each bar would be. A heavy dot is shown on Fig. 13*a* for each of the seven points so located. A smooth curve was drawn through these points to obtain the frequency curve shown on the figure.

This is a normal, symmetrical, frequency curve for a theoretical distribution of data that obey the laws of probability. The vertical distance from the base line to the curve gives the frequency of occurrence of the number of heads represented by the corresponding number on the base line. For example, to determine how many times exactly 5 heads will be thrown out of 64 tosses of 6 coins, project up to the curve from 5 on the base line, and from this point on the curve project horizontally to the left and read 6 on the vertical scale.

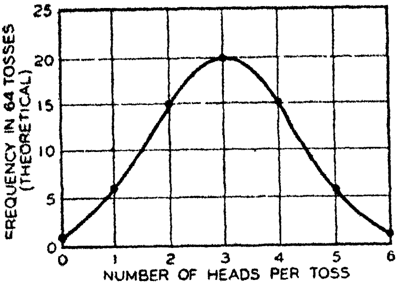

Fig. 13a.

The frequency curve of Fig. 13b has been drawn with a vertical base. This curve has been superimposed on a tally record taken from Table 11a in order to exhibit how the frequency curve actually represents frequency of occurrence of the variables.

To read this curve we project horizontally from the base line, now along the vertical scale, to the curve, then down and read the frequency on the horizontal scale.

The point farthest from the base line will be called a "high point" or a "peak" whether the base line is horizontal or vertical. If the distribution is a normal, symmetrical one, the peak will correspond to the arithmetic mean value of all the variables of the frequency distribution represented. The line perpendicular to the base line and passing through the peak will be called the center line or the average line for the curve. This line may be denoted by ℄.

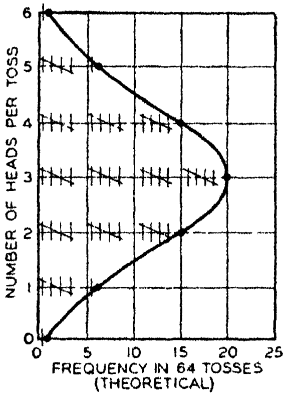

Fig. 13b.

14. Frequency Distribution for a Controlled Process

The variables of a controlled manufacturing process obey the laws of chance that govern in the case of tossing coins, and will have a very similar frequency curve. There is, however, one very important difference. With one toss of 6 coins it is impossible to get more than 6 or less than 0 heads, while in the manufacturing process it is possible to have values outside of either specification limit. This could happen by chance or it could happen if something went wrong with the process. It will be seen later that, with a controlled process, a value could fall outside of limits for the frequency curve by chance only about 3 times out of 1,000. Since this chance is so small it is always wise to assume that if a value goes outside of limits it was caused by something going wrong with the process. Hence in each case a search should be made for an assignable cause of trouble.

The frequency distribution for a controlled manufacturing process is similar to those for variables from nature that are controlled by laws of chance. Heights and weights of men, as well as their shoe sizes and collar sizes, are examples of variables from nature that have normal frequency distributions. These, of course, must be selected from groups that possess common general characteristics. Measurements of giants and dwarfs should not appear in the same frequency distribution.

In Art. 15 we shall give an application to heights of men, and in Art. 16 we shall give one from a manufacturing process.

15. Heights of Men

In order to obtain data from which to make a statistical study of the heights of a large number of men it is unnecessary to measure the heights of every man in the group. A sample may be selected provided it is chosen in an unbiased manner and is large enough so that the average, the general distribution, and other important features of the sample are truly representative of the whole group.

Suppose that we want a sample of 500 from a group of approximately 2,000 and that we want the heights of men in this sample to be representative of the heights of all in the large group. Obviously we could not get satisfactory data by lining the men up according to heights from the tallest to the shortest and selecting the first 500, the last 500, or any 500 in consecutive order among the men so arranged. Some random form of selection must be used, a form which gives each one an opportunity to be chosen. Since a truly random selection is difficult to obtain, it is customary to use some method that will give a sufficiently representative sample for it to be considered as a random one. For example, if in the present illustration the men were arranged in any manner[1] and every fourth man were taken until 500 had been selected the distribution of heights of the men in this sample should correspond very closely to that for the total group.

The heights and the weights of 500 male students at the University of Cincinnati are given in Table 15*a*. The data for this table were obtained from the records of the Health Service at the University for a year prior to the Second World War when approximately 2,600 male students were in attendance. The method of selection followed was to take every fifth card from an alphabetical arrangement of the students' records. This selection should be satisfactorily random.

Next we must obtain a frequency table for the data of Table 15*a*, but first we must give some attention to the general question of subgrouping the data.

In courses on statistical methods much attention is given the question of subdividing data into groups or classes. Of these questions we shall be interested only in the one regarding an appropriate number of groups into which to subdivide the data. That is a difficult question to answer because there is no fixed rule to follow. It requires practice and good judgment to know what is a good selection in any particular case. As a general rule it is unwise to have fewer than 6 or more than 20 classes. Fortunately in much of the work with a manufacturing process the unit of measurement will suggest a good division for class intervals. With the case of tossing coins it was logical to take 7 classes according to the possible number of heads that could appear with any toss of the 6 coins. The number of classes for the coin problem is shown by the 7 entries of the frequency tables, the 7 bars of the bar diagrams, and the 7 points of the frequency curves for Tables 11*a*, *b*, *c*, and Figs. 12*a*, *b*, and 13*a*, *b*.

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[1] In any manner except one determined by the characteristic in which we are interested. Obviously if the men were arranged so that every fourth man was chosen from the 500 shortest men then the sample would not be a satisfactory representation of the total group.

TABLE 15a.—HEIGHTS AND WEIGHTS OF 500 STUDENTS

Height	Weight	Height	Weight	Height	Weight	Height	Weight
69.5	158.0	65.5	126.0	62.6	105.0	69.5	149.0
67.0	125.2	68.7	138.0	68.4	155.7	71.0	148.0
69.1	156.0	66.6	112.5	66.7	150.0	68.2	144.0
70.5	169.0	72.5	147.0	67.2	120.0	69.5	154.5
67.2	138.0	68.7	157.0	69.4	138.7	66.0	132.0
69.0	134.0	60.2	164.7	71.0	170.5	66.7	130.0
72.5	148.5	69.0	164.0	70.0	163.0	71.3	174.0
69.7	139.0	65.2	131.5	69.0	127.5	73.0	157.5
68.5	125.0	68.1	175.5	67.0	131.0	70.0	145:5
68.7	138.0	69.5	129.0	65.5	150.5	74.0	166.0
67.3	128.7	68.2	171.5	68.5	136.0	68.5	133.0
70.5	143.0	73.0	161.5	70.2	159.0	72.1	146.7
70.0	132.0	68.5	139.0	67.0	137.7	69.3	158.1
68.0	153.0	66.0	131.0	64.5	123.0	69.0	144.5
69.0	141.5	69.0	143.0	72.0	148.5	66.3	124.5
69.5	144.0	68.1	128 5	66.3	122.5	67.5	127.0
66.0	129.5	70.7	143.0	69.5	122.5	66.6	124.7
72.2	172 0	68.0	137 0	62.5	125.5	69.7	136.2
67.0	137.5	64.4	125.2	72.0	142.5	68.5	125.0
71.0	142.5	69.0	159.5	69.7	130.0	71.3	161.0
69.5	148.0	70.2	134.0	71.0	141.5	68.0	147.0
66.7	111.5	67.2	154.5	68.5	135.5	67.0	149.7
71.0	157.0	67.7	160.0	69.5	137.0	70.0	122.2
66.0	176.2	69.5	141.0	65.6	126.0	69.6	136.0
64.4	147.0	72.0	171.0	67.4	156.0	62.5	121.5
75.7	175.5	71.0	144.0	72.5	160.0	70.0	122.0
68.1	123.0	67.5	150.0	67.9	131.0	70.2	149.0
66.5	118.7	67.3	145.0	68.5	147.5	65.2	142.0
70.0	149.0	66.8	125.0	65.7	131.5	71.0	146.0
71.2	156.0	66.0	141.5	72.5	156.0	72.0	137.0
67.0	143.0	66.7	141.0	69.5	136.2	69.0	131.0
65.0	138.0	64.0	112.5	69.6	170.0	71.5	143.5
71.7	148.0	67.5	143.0	67.6	141.7	70.0	139.0
68.6	122.5	71.6	146.2	67.4	127.0	66.0	125.7
74.2	152.0	70.7	148.2	71.5	135.5	67.0	134.0

TABLE 15a.—HEIGHTS AND WEIGHTS OF 500 STUDENTS.—(*Continued*)

Height	Weight	Height	Weight	Height	Weight	Height	Weight
66.5	117.0	67.8	129.5	66.0	124.0	69.0	155.0
68.0	138.5	67.7	134.5	64.5	121.5	69.0	145.0
70.6	144.0	67.4	146.0	65.5	136.0	67.0	132.0
69.0	139.0	67.5	129.5	70.0	121.2	70.5	157.0
71.7	140.5	66.2	120.0	69.5	139.0	69.2	126.0
68.0	138.0	68.7	162.0	72.6	167.2	67.0	144.0
67.0	112.0	69.9	134.1	65.1	124.0	68.5	126.2
67.0	141.5	69.1	115.0	71.7	149.0	69.7	145.0
71.5	162.0	72.0	148.0	68.0	133.2	66.7	106.0
68.0	130.5	66.5	104.0	64.5	113.0	65.4	126.5
73.0	172.0	66.0	115.0	67.5	134.2	67.0	136.0
71.0	158.5	67.0	128.0	69.7	147.0	73.2	143.0
70.7	153.0	69.0	143.2	66.7	123.0	67.9	133.0
68.0	157.0	68.0	110.0	73.0	186.0	69.0	180.5
68.0	128.5	69.0	139.5	69.3	139.5	67.0	147.0
65.6	120.7	71.5	159.5	68.4	147.7	72.2	155.5
72.0	161.0	72.2	140.5	71.5	129.5	69.0	126.0
71.0	133.0	68.0	122.0	69.0	151.5	71.6	175.5
70.2	141.5	71.6	137.0	69.2	148.0	70.5	138.0
67.2	128.0	68.5	145.7	67.5	142.0	72.4	138.2
68.0	147.0	70.0	145.5	64.0	104.0	69.2	148.1
67.1	151.7	65.0	122.5	67.5	163.0	65.2	130.0
73.2	163.5	67.0	146.5	67.5	159.0	68.5	148.5
66.0	120.5	71.1	151.2	69.0	146.5	69.0	143.0
68.2	126.0	67.0	146.0	70.2	155.0	67.8	130.0
68.0	138.0	70.0	160.5	70.2	162.0	67.2	125.0
66.5	135.0	66.2	138.0	69.4	131.0	68.7	135.0
64.0	118.0	67.7	126.0	65.0	110.0	68.2	147.5
69.0	142.0	70.2	141.5	66.0	128.0	68.5	114.7
68.2	142.0	72.7	164.0	64.6	118.2	69.2	133.5
68.1	132.0	70.2	134.5	71.2	171.0	71.0	174.5
64.6	128.5	71.4	141.0	68.0	130.0	64.0	125.7
67.0	146.0	65.5	138.0	74.0	166.0	71.5	181.0
66.7	146.2	66.0	118.0	72.5	153.2	65.2	134.0
69.2	147.5	72.5	175.0	69.9	127.5	65.5	134.0

TABLE 15*a*.—HEIGHTS AND WEIGHTS OF 500 STUDENTS.—(*Continued*)

Height	Weight	Height	Weight	Height	Weight	Height	Weight
70.0	165.0	66.5	143.0	71.3	189.7	68.3	126.2
68.2	153.0	67.0	156.0	70.0	137.2	62.4	118.2
70.2	128.5	69.0	147.0	67.6	142.5	69.0	142.0
65.0	140.0	70.0	142.0	70.7	152.0	71.0	139.0
69.5	126.0	70.2	141.5	67.2	146.0	67.2	128.0
64.8	135.0	68.6	133.0	70.7	151.5	70.5	126.0
65.0	124.0	63.0	131.0	66.0	110.0	70.2	191.0
70.5	146.0	70.4	165.5	69.7	154.0	70.4	137.5
68.2	128.5	65.5	113.0	69.3	130.5	71.0	154.0
68.8	138.7	65.4	140.0	68.2	144.5	71.5	166.5
68.3	145.7	72.7	135.0	69.5	140.0	71.0	154.0
75.0	149.2	66.5	127.0	69.4	149.5	70.7	159.0
72.5	154.0	69.2	147.5	63.0	131.5	68.7	129.5
69.0	139.0	66.4	127.7	67.7	156.0	72.0	170.0
69.2	131.0	71.6	158.5	68.2	127.0	67.0	121.0
68.5	126.0	71.5	158.5	62.7	111.0	67.5	148.0
68.0	149.0	69.6	139.0	67.0	132.2	66.0	123.0
70.5	152.0	71.7	186.0	67.4	192.0	69.7	137.0
71.0	136.0	69.3	127.0	72.6	153.0	68.2	140.0
71.0	132.7	65.5	152.0	67.4	147.0	66.6	120.7
67.2	125.0	70.2	132.7	66.4	133.0	71.0	138.2
70.5	148.0	73.5	175.0	69.7	170.0	70.5	173.5
68.5	138.5	69.7	135.0	72.0	148.5	73.5	146.0
67.5	125.0	69.0	142.5	66.0	154.5	66.8	140.0
68.2	165.0	70.4	129.5	71.7	150.5	69.5	147.5
71.0	199.5	71.0	136.5	68.2	125.0	70.0	146.0
68.5	139.0	74.0	153.0	68.5	163.5	66.3	138.2
73.9	172.0	70.9	127.5	67.5	141.0	68.5	152.0
71.0	150.0	71.0	143.0	64.0	122.0	67.5	148.5
68.0	138.5	72.0	148.5	64.0	126.5	68.5	152.0
70.0	141.0	73.6	153.0	67.7	139.0	77.6	168.2
67.6	150.0	67.0	149.0	63.4	115.7	69.5	139.2
71.0	161.5	69.0	138.7	66.0	119.5	68.5	140.0
67.5	136.0	68.5	162.0	68.0	133.5	70.0	156.5
67.7	130.2	65.5	113.0	68.5	145.0	65.5	126.5

TABLE 15a.—HEIGHTS AND WEIGHTS OF 500 STUDENTS.—(*Continued*)

Height	Weight	Height	Weight	Height	Weight	Height	Weight
65.9	122.5	68.5	148.5	73.4	150.0	66.2	119.0
68.5	132.5	66.0	131.0	72.7	158.0	71.3	169.0
65.2	158.0	71.1	150.5	67.2	137.0	68.7	145.0
71.0	150.5	68.2	144.5	71.5	160.5	73.0	147.5
69.3	158.5	70.5	142.5	64.5	130.0	72.0	144.0
66.0	114.0	70.5	130.0	68.7	170.7	73.1	155.0
70.0	138.0	70.0	158.0	69.9	135.0	70.5	153.0
62.1	107.0	65.0	116.2	70.9	151.7	71.8	155.0
70.6	165.0	71.2	143.5	70.5	149.5	67.2	147.0
71.0	159.0	71.5	160.0	65.5	135.5	68.0	137.0
69.0	131.7	66.5	152.0	69.5	107.0	69.5	154.0
68.4	149.0	66.7	136.5	67.5	134.5	66.0	124.5
66.0	165.2	67.9	127.7	69.0	156.0	67.5	154.2
63.5	116.0	67.2	133.0	64.5	120.0	66.7	145.0
72.7	185.0	73.5	184.0	70.7	156.7	71.0	148.0
68.5	126.5	69.7	158.5	68.5	181.0	66.0	116.0
71.5	137.0	65.5	101.7	68.5	145.0	66.2	139.0
67.0	128.0	69.0	134.5	64.0	106.0	70.7	145.0
68.0	188.5	65.7	132.7	70.3	155.5	71.5	147.0
69.0	152.0	70.7	169.5	69.9	166.5	69.3	148.0

Let us now give detailed attention to subdividing the data o Table 15*a* for heights of students into groups or classes. On th cards of the Health Service records the heights of students wer given to the nearest tenth of an inch, as they appear in Table 15*a* As the heights range from approximately 60 inches to 78 inches a tenth of an inch per class interval would give far too man classes. Tables 15*b* and 15*c* were made with 19 classes by usin the heights to the nearest inch. Thus 68.1 inches, 68.2 inches 68.3 inches, and 68.4 inches would all be tallied as 68 inches while 68.6 inches, 68.7 inches, 68.8 inches, and 68.9 inches woul be taken as 69 inches. The half-inch was increased to the nex inch, so that 68.5 inches would also appear as 69 inches.

The frequency distribution, including the tally record, for th heights of the first 100 students listed in Table 15*a* is given i Table 15*b*. This table is given to illustrate a sample that is no

TABLE 15*b*.—FREQUENCY DISTRIBUTION FOR HEIGHTS OF 100 STUDENTS

Height in inches X	Frequency of occurrence	
	Tally record	Frequency f
78		0
77		0
76	/	1
75		0
74	/	1
73	𝍸	5
72	𝍸 /	6
71	𝍸 𝍸	10
70	𝍸 𝍸 ////	14
69	𝍸 𝍸 𝍸 //	17
68	𝍸 𝍸 /	11
67	𝍸 𝍸 𝍸 //	17
66	𝍸 ////	9
65	///	3
64	///	3
63	//	2
62		0
61		0
60	/	1
	Total	100

large enough to represent the group from which it was taken. However, even this sample gives a fairly satisfactory distribution from which to obtain important features of the larger group. If you should make an attempt to superimpose a frequency curve on the tally record, as was done with Fig. 13*b*, you would have to decide whether to show two peaks on the curve, one for 67 inches and the other for 69 inches, or to draw a smooth curve with a peak between 68 inches and 69 inches as indicated by the trend on either side of these two dimensions. The low frequency for heights of 68 inches is unnatural and in Table 15*c*, where the data for all 500 students are given, we shall see that the deficiency in frequency for this height does not hold for the larger group.

The frequency distribution, without the tally record, of the heights of 500 students is given in Table 15*c*.

TABLE 15*c*.—FREQUENCY DISTRIBUTION HEIGHTS OF 500 STUDENTS

Height, inches X	Frequency f
78	1
77	0
76	1
75	1
74	9
73	21
72	37
71	60
70	68
69	83
68	70
67	60
66	47
65	22
64	10
63	7
62	2
61	0
60	1
Total........	500

The frequency curve for the data of Table 15*c* has been drawn on Fig. 15 with a vertical base line. From this curve the distribution is seen to be quite normal and symmetrical. The reader may better appreciate the meaning of a frequency curve if he will try to picture Fig. 15 as representing 500 men standing on a field. To do this let the base line represent a wall in front of which the men are lined up in rows perpendicular to the wall. Let each dot on the figure represent a man, the dots in any row to indicate men whose height, to the nearest inch, is shown at the left end of the row. If you could now look down from a plane over the field, the men would appear as the dots do on the figure. If a curve should be chalked on the field passing through, or approximately through, the position of the man at the right end of each row, the curve would take the general form shown in Fig. 15.

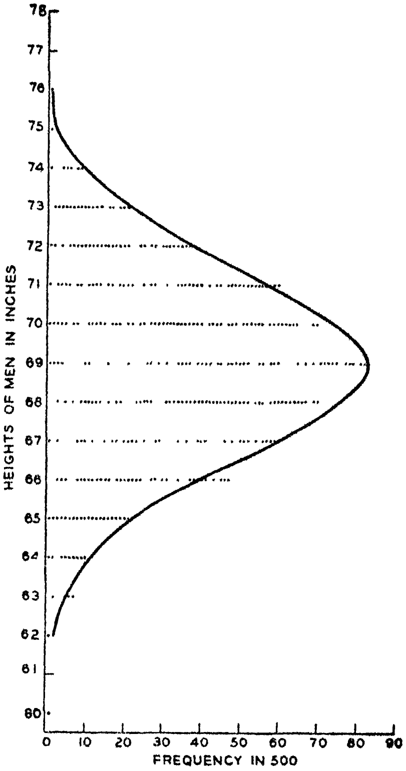

FIG 15.

There should be no attempt to extend the curve below 62 inches or above 75 inches. However, the curve might be smoothed out from 74 inches to 76 inches as though there had been three or four students with heights of 75 inches. From symmetry, we should expect a few more with heights of 75 inches and perhaps not quite so many with heights of 63 inches. A curve for 1,000 students would probably show these heights with more nearly the same frequencies.

One single point at 60 inches and another at 78 inches correspond to heights that occur very rarely—once or twice in 1,000. These represent cases that could not occur with the demonstration of tossing coins, but that are expected to occur (though very rarely) in any particular set of variables from nature. Such cases could occur by chance (though again very rarely) with a controlled manufacturing process, but the chance of this happening with a controlled process is so small that when it does happen it is wise to assume that something has gone wrong with the process and to start looking for an assignable cause of trouble.

In courses on statistical methods it is shown that about 99.73 per cent of the variables of a normal distribution will fall between limits that may be determined from the distribution of these variables in a representative sample chosen from the total distribution. The limits for the heights of students, computed for the frequency distribution of Table 15*c* by formulas [1] of statistics, are 61.4 inches and 76.4 inches. These limits are determined from the total frequency distribution of the sample of 500 and not from the spread between the smallest and the largest entries of the table. If the sample of 500 is truly representative of the larger group from which it was selected, then out of each 1,000 of the larger group not more than a total of 3 should be found who were shorter than 61.4 inches or taller than 76.4 inches.

16. Diameters of Bushings

To obtain data for a frequency distribution of a manufactured item, if the items are taken from the production line at regular intervals, one at a time, or a few at a time (not over 10, but preferably 4, 5, or 6), then the selection would be sufficiently random to be used as representing the lot. The outside diameters of bushings, taken from the production line 5 at a time, at approximately every 30 minutes, are recorded in Table 16*a*. The bushings were being made to meet specifications for an outside diameter of 2 inches ± 0.004 inch, and the process was under control. Measurements were recorded to the nearest thousandths of an inch.

It should be noted that actual measurements, to the nearest thousandth of an inch, were recorded. In order to establish whether or not a process is under control it is not enough to have the items satisfy the go, no-go-gage, but actual measurements are needed.

The frequency distribution, without tally, of the data of Table 16*a* is given in Table 16*b*. For the sake of comparison, a good theoretical symmetrical distribution is given in Table 16*c*.

A frequency distribution based on a total of only 150 items is hardly adequate to establish the distribution for the lot from

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[1] Formulas will be given in Chap. V.

TABLE 16*a*.—OUTSIDE DIAMETERS OF 150 BUSHINGS

(*Measurements to the nearest thousandth of an inch*)

2.003	2.000	2.001	2.001	1.999	2.002
2.001	2.000	2.000	2.004	2.000	2.000
1.997	1.999	2.002	1.998	2.000	1.999
2.002	2.002	2.000	2.000	2.002	2.000
1.999	1.998	2.000	1.999	2.001	1.999
2.000	2.000	2.000	1.998	2.001	2.001
1.998	1.998	1.997	2.000	2.003	2.000
2.001	2.002	1.999	1.999	2.000	2.000
1.999	2.000	1.998	2.004	2.000	2.003
1.999	2.000	2.000	2.001	2.000	2.001
1.997	1.999	2.001	1.999	2.001	1.999
1.998	2.002	1.999	1.999	1.998	1.998
1.997	1.999	2.000	2.002	1.999	2.003
1.998	2.001	2.003	2.001	1.998	2.001
2.000	1.998	2.000	1.999	1.999	2.001
2.001	2.001	2.000	1.998	2.001	2.000
2.001	2.000	2.000	2.001	2.001	2.000
1.998	1.999	1.999	2.002	2.000	1.997
2.001	2.000	1.999	2.000	1.998	2.000
1.999	2.002	2.003	2.000	2.000	2.001
2.001	2.001	1.998	1.999	2.002	2.001
2.001	2.000	2.001	1.999	1.999	2.000
2.001	1.999	1.997	2.002	1.999	1.998
1.999	1.998	2.000	2.002	1.996	2 000
2.001	2.002	2.000	1.999	2.001	2.001

TABLE 16*b*.—FREQUENCY DISTRIBUTION FOR O.D. OF 150 BUSHINGS

Diameter, inches X	Frequency in 150 f
2.004	2
2.003	6
2.002	14
2.001	32
2.000	41
1.999	30
1.998	18
1.997	6
1.996	1
Total........	150

TABLE 16*c*.—A THEORETICAL DISTRIBUTION FOR O.D. OF 150 BUSHINGS

Diameter, inches X	Frequency in 150 f
2.004	1
2.003	5
2.002	16
2.001	33
2.000	40
1.999	33
1.998	16
1.997	5
1.996	1
Total........	150

which the items were chosen. However, it is customary to use measurements of from 100 to 200 items as a basis for a preliminary analysis of the state of control of a process. If the process is under satisfactory control then measurements of 150 items will furnish a reliably accurate representation of the condition of control of the lot from which these items were chosen. In the present illustration we see that the distribution of Table 16*b* is not too different from the theoretical distribution of Table 16*c*.

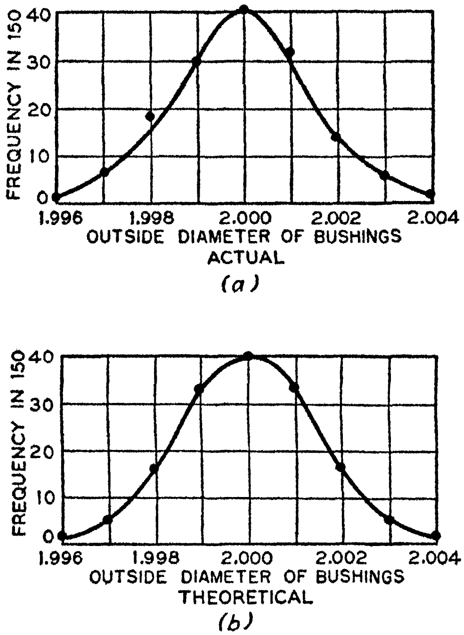

Fig. 16.

The frequency curves for the data of Tables 16*b* and 16*c* are shown on Figs. 16*a* and 16*b*, respectively. It should be noted that in drawing the curve of Fig. 16*a* the plotted points for diameters of 1.998 inches and 2.001 inches were missed. The resulting approximately symmetrical curve should be satisfactorily representative of all the outside diameters of the bushings of the lot from which these were chosen. If not, then this fact will become evident on the control chart, which will be based on the measurements of the 150 bushings. How this is done will become clear after studying the presentation of control charts in Chap. III, where a chart will be made for the data of Table 16*b*.

17. Danger Limits

Limits within which practically all diameters of bushings of the process from which Table 16*a* was prepared, as calculated by formulas of statistics, are 1.9955 inches and 2.0045 inches. This means that, if nothing goes wrong with the process, approximately only 3 out of 1,000 should be produced which, by chance, will have diameters less than 1.9955 inches or greater than 2.0045 inches, or, as measurements were to the nearest thousandth of an inch, less than 1.9950 inches or greater than 2.0050 inches. We shall not present the formulas of statistics for calculating these limits in this chapter,[1] but in Chap. III we shall give substitute, but very much simpler, formulas that are commonly used in applications of statistical methods in the control of quality. Furthermore we shall not now designate these limits by the term used for them in courses on statistical methods, but shall call them "danger limits." In Chap. V this term will be identified with the one commonly used.

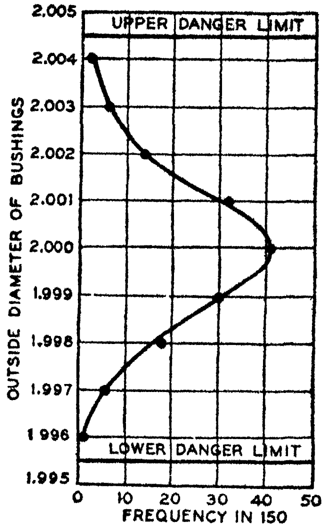

FIG. 17.

The frequency curve for the data of Table 16*b* for the 150 bushings, which was shown on Fig. 16*a* with a horizontal base line, has been redrawn on Fig. 17 with the base line vertical. On this figure the danger limits have been located and identified.

Danger limits are limits within which individual items should fall. With a frequency distribution which is as normal as that shown by the curve of Fig. 17, and with the danger limits as far from the ends of the curve as they are, it should be obvious that there is very slight possibility that a bushing will be produced with a diameter as small as 1.995 inches or as large as 2.005 inches.

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[1] Formulas will be given in Chap. V.

But the control of quality cannot be fully judged by the appearance of the frequency curve. We shall see later that the frequency curve, when properly interpreted, gives a very valuable indication of control. One aid to this proper interpretation will be the control limits which will be presented in the next chapter.

Since a manufacturing process is being watched with great care, it would be more unusual for a bushing to be made with a diameter that would fall outside the danger limits for the frequency curve for the bushings than it would be for a student to be found whose height was outside the limits for the frequency curve of Fig. 15 for heights of students. Danger limits were not shown on Fig. 15, but the limits 61.4 inches and 76.6 inches given in Art. 15, are danger limits for the frequency distribution of heights of students. With both the heights of students and the diameters of bushings the limits are set by the distribution of the variables of the respective frequency tables. The two sets of variables obey the same laws of chance, but with the manufactured product a more definite condition of control can be maintained.

18. Chart for Individual Items

Points corresponding to diameters of individual bushings have been plotted on Fig. 18. Points were plotted in groups of five for the diameters recorded in Table 16*a*, and were selected from that table in order reading down the columns, starting with the first column. The danger limits of Fig. 17 have also been drawn on Fig. 18.

Much important information can be gained by studying Fig. 18. It is a useful preliminary form of chart for observing the behavior of the variables until enough data are available on which to consider the condition of control for the process. However, the question of control cannot be determined from Fig. 18 any more fully than it could be determined from Fig. 17. It is true that, so long as all points of Fig. 18 fall within the upper and the lower danger limits, and these limits are within specification limits, the bushings will meet specification requirements. But there is nothing yet to indicate whether or not the bushings will continue to meet those requirements. Control charts will give us a definite basis for judging whether or not the process is in control. If the control chart indicates that the process is under satisfactory control then we can be sure that bushings as made will continue to meet specification requirements so long as the controlled condition is maintained. If something goes wrong, it should soon be detected from the control chart, and an investigation should be made immediately to determine the cause of the trouble.

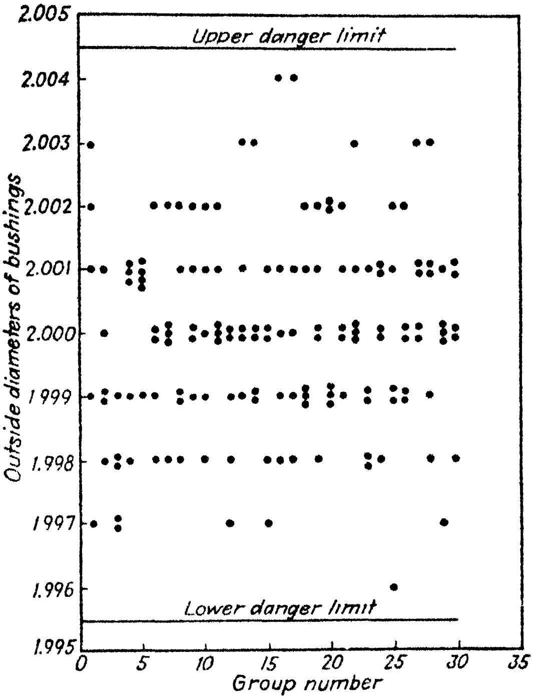


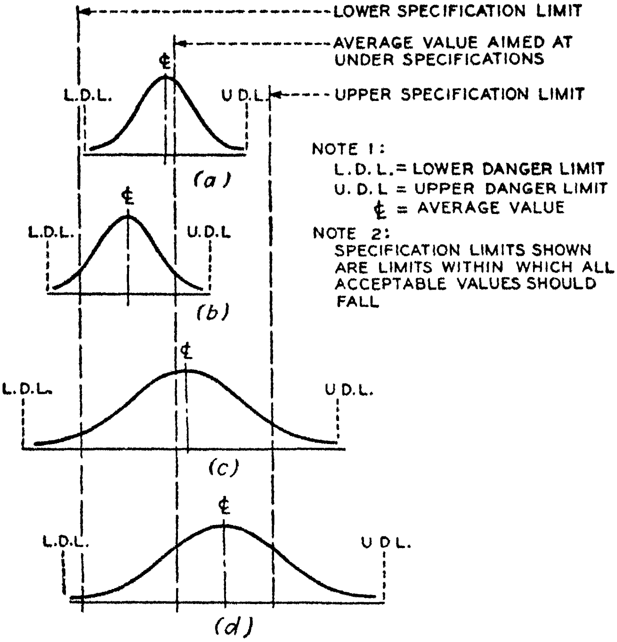
FIG. 18.

In the next chapter control charts will be introduced and a control chart will be drawn for the data on outside diameters of bushings of the present illustration.

19. Use of Frequency Curves

A frequency curve is important not only in providing a basis for control charts, but also in having an intrinsic worth of its own. The smoothness and the general appearance of the curve, the condition of symmetry, the location of the axis of symmetry, the dispersion of the points about the axis of symmetry, the spread on the base line, and the height of the curve at the highest point, are some of the features that furnish valuable information regarding the variables of the distribution. With a carefully drawn curve it is possible to identify an unsatisfactory average and also a spread that would indicate a high percentage of scrap.

It is often easier to explain important matters of control to someone who is unfamiliar with statistical quality control by


LOWER SPECIFICATION LIMIT
AVERAGE VALUE AIMED AT UNDER SPECIFICATIONS
UPPER SPECIFICATION LIMIT
℄
L.D.L.
U.D.L.
(a)
(b)
(c)
(d)

NOTE 1:
L.D.L. = LOWER DANGER LIMIT
U.D.L = UPPER DANGER LIMIT
℄ = AVERAGE VALUE

NOTE 2:
SPECIFICATION LIMITS SHOWN ARE LIMITS WITHIN WHICH ALL ACCEPTABLE VALUES SHOULD FALL

Fig. 19.

means of a frequency curve than by using a control chart. A satisfactory condition of control is represented by the curve of Fig. 19*a*, while three unsatisfactory conditions are shown in Figs. 19*b*, 19*c*, and 19*d*.

The quality control engineer should be able to convince management that the condition represented by the curve of Fig. 19*a* is satisfactory, but that some action should be taken to improve the process in each of the cases represented by Figs. 19*b*, 19*c*, and 19*d*. He should be able to show that a small percentage of scrap would be produced on the low side in the process represented by Fig. 19*b*, a large percentage on both the high and the low side for the curve of Fig. 19*c*, and a still larger percentage of scrap for the process represented by Fig. 19*d*. He could demonstrate that, for the process given by Fig. 19*b*, a correction could be made very readily, perhaps by a slight tool or other adjustment by the operator or supervisor, and the process be brought under satisfactory control. He should also be able to show that a condition such as represented by the wide spread in Fig. 19*c* and 19*d* is more serious and if found then one of three things must be done: (1) Change the process so that the spread will be brought within specification limits; or (2) Change the specification limits to conform with what the process can do; or, if neither of these can be done, (3) Admit that the specification requirements cannot be

TABLE 19.—LENGTHS OF 150 BUSHINGS

(Measurements to the nearest thousandth of an inch)

1.001	1.000	1.002	0.998	0.997	0.997
0.999	0.999	1.000	0.998	1.001	0.999
1.000	1.000	0.999	1.001	0.999	0.998
1.002	0.999	0.998	1.000	0.999	1.002
0.999	0.998	0.998	1.001	1.002	1.000
0.998	1.001	1.002	1.001	1.001	1.003
1.001	1.001	1.001	0.998	0.999	0.999
0.999	1.001	1.001	1.000	1.000	1.003
1.000	1.001	1.000	1.000	1.000	1.002
1.000	1.000	1.000	1.002	1.000	0.998
0.999	1.000	1.000	1.002	1.000	1.000
1.000	0.999	1.001	0.996	1.001	0.998
1.002	1.001	0.999	0.998	0.999	1.001
0.999	1.002	1.003	1.001	1.001	0.998
1.003	1.001	0.999	1.001	0.999	1.002
1.001	0.999	1.000	0.999	0.999	1.000
0.997	1.001	0.999	1.002	1.002	0.998
0.999	0.997	1.001	0.997	1.000	1.000
1.000	1.000	1.001	1.002	1.000	0.999
0.999	1.000	1.000	1.000	1.002	0.999
1.000	1.000	1.000	1.000	0.999	1.000
1.003	1.003	1.002	1.000	0.998	1.001
1.004	0.999	0.999	0.998	1.001	1.003
1.002	1.000	1.001	1.003	0.999	0.997
0.996	0.999	1.001	1.000	1.004	1.001

met economically with facilities that are available for the process, and that scrap will be produced and must be found by inspection. If he can show management that an average saving of a specified amount in the cost of each item will result for each little division that the curve in Fig. 19*b* is moved toward the right, or for each of these divisions that the spread is reduced in the other cases, then he should have no trouble in securing authority for making necessary improvements.

EXERCISES

1. Prepare a frequency distribution table, including a tally record, for the weights of students shown in Table 15*a*, and draw a frequency curve. Use the weights to the nearest 5 pounds starting with 100 pounds.

2. Prepare a frequency distribution table, including tally record, for the lengths of the 150 bushings given in Table 19, and draw a frequency curve.

The bushings were taken, in the order of production, from a production line of a manufacturing process that was under satisfactory control, and each bushing was measured to the nearest thousandth of an inch. The lengths were to meet specifications of 1 inch ± 0.004 inch.

CHAPTER III

CONTROL CHARTS FOR AVERAGES AND RANGES

20. Method of Inspection

There are several different control charts which we shall present in this chapter and the chapters that follow. The charts may be considered in two general classifications depending upon whether inspection is for variables or for attributes.

a. Inspecting for Variables.—When inspection is for variables, the items are taken from the production line, in order of production, at regular intervals, in small samples, or groups, usually of 4, 5, or 6. Each item of the sample is carefully measured and the actual measurements are recorded to the nearest fraction of a unit used with each particular manufacturing process. The resulting control chart gives a reliably accurate indication of the quality of the product. When a condition of satisfactory control of quality has been attained, the chart can be used to maintain control during production.

b. Inspecting for Attributes.—When inspection is for attributes, the items are checked only to see whether or not they meet specification requirements. If an item meets these requirements it is accepted, otherwise it is rejected. Inspection may be visual to see whether there are flaws that make an item unacceptable, or the item may be checked by some standard to see if it does or does not function as intended. For a machined part to be acceptable it is only necessary that it pass the go, no-go-gage. Inspection is 100 per cent for all items, though frequently a large sample is taken from a very large lot and the sample inspected 100 per cent. More often than not a lot inspected represents the output for a day. The control charts that result from inspection by attributes give quite satisfactory indication of the quality of the product. These charts are very simple to make, and can be readily explained and understood, but are not so useful for controlling quality during production as the charts obtained through inspection by variables.

In the present chapter, we shall explain the methods of constructing and interpreting charts that are based on data obtained from inspecting by variables. The charts that we shall present are *charts for averages* and *charts for ranges*. Different steps in the construction of a chart will be considered in general terms and will be illustrated by applying the steps to the problem with the bushings that was introduced in Chap. II.

21. Preliminary Considerations

Whenever a control chart is to be made in a factory there are certain preliminary matters that must be considered and general procedures that must be decided upon. The reader who has done no work with control charts is unfamiliar with these preliminary matters. Therefore at this time we should mention a few things that affect the charts in order that the beginner may observe them as he sees the charts developed, and may notice their influence upon the work.

In the first place the type of chart must be selected, and the method of inspection decided upon. These two are so closely related that it is difficult to say whether the method of inspection fixes the type of chart or the type of chart fixes the method of inspection. It will also be necessary to decide for what product and for what characteristic of that product the control chart is to be made. For example, if manufacturing an engine, the product might be a pinion and the characteristic might be the pitch diameter, or the outside diameter of a shank, or the over-all length, or any other important dimension. Or if manufacturing chemicals, the product could be a certain compound and the characteristic could be the percentage of a certain element which it contained. Illustrations could be given that would include every commodity that is manufactured, for with each type of manufacturing we could select many products and several characteristics for each product. With each characteristic selected a control chart would be useful in making the selected characteristic so that it would meet specifications.

Let us consider some general matters that come up for preliminary consideration. It must be decided how often to take a sample, and what size sample should be taken. Specification requirements must be noted. The units of measurement must be chosen, and the method of making and recording measurements must be decided upon. A form on which to record the data must be designed or selected. The control chart form must be prepared, with units and scales, and, at the appropriate time, control limits must be added. It must be decided who is to inspect the items, who is to record the data, who is to plot the points, and who is to have general responsibility for the chart. In a plant many of these matters are permanently decided through the organization of the quality control department in the plant.

The matters mentioned are not the only ones to be considered before starting a control chart, but they cover the most important ones. Other may be noted as the work progresses.

In the next article, we shall make a chart for averages and a chart for ranges for the outside diameter of the bushings of Table 16*a*. Samples of 5 were taken approximately every 30 minutes, the diameter of each bushing was measured to the nearest thousandth of an inch, and the measurements were recorded in order, reading down the first column, then the second column, and so on through the table. The samples were not taken at exactly 30-minute intervals in order not to affect the randomness of the selections by having the operator more careful at the instant the inspection was to be made.

22. Charts for Averages and for Ranges

Let us assume that we started our control chart record and calculations when we started to collect the data of Table 16*a* and that we carried the record of the measurements in the spaces provided on the form of Table 22 simultaneously with that of Table 16*a*. Then let us outline the various steps as we proceed to make computations, plot points, locate limits, and construct and discuss the charts.

a. Preparing the Record Form; Sample Size.—The record form that we have used can be identified by imagining that none of the entries have been made in Table 22. We first fill in the blanks at the top of this form and then are ready to make entries for each sample as taken, and to compute and record averages and ranges.

The form shown can be used for a sample of 2, 3, 4, or 5 items. On the basis of the formulas to be used for calculating control limits any sample size of 2 to 10 is allowable, but a sample size of 4 or 5 is usually more satisfactory than is any one of the others. We shall use a sample of 5 in the present illustration and shall use 4 with the illustration of the next article.

b. Form for the Control Charts.—The form for the control charts may be laid out on a sheet of standard cross-section paper, or on a sheet designed for the purpose. The sheet on which Fig. 22*a* has been drawn was designed for showing a chart for averages and a chart for ranges lined up together for the same data. Both charts are kept just as long as either may give important information that would not be evident from the other. Sometimes the condition of control can be determined from one chart alone, and in these cases it is satisfactory to discontinue plotting the chart that is not needed. The chart for averages is perhaps used more frequently alone than is the chart for ranges.

With each chart the horizontal scale should indicate the sample number, and the vertical scale should indicate the appropriate magnitudes for the items measured. If the chart extends over more than one day, an additional horizontal scale could be included to show the dates. Since our illustration will include 30 samples there must be at least 30 divisions on the horizontal scales.

For the present illustration the vertical scales range from 1.9940 to 2.0060 on the chart for averages and from 0 to 0.010 on the chart for ranges. On the chart for averages the desired normal value of the item should be centered in the space allowed for this chart and the scale should extend the same distance above and below this central value. It is necessary to extend this scale only far enough toward zero to include all values that might arise in the particular application. In the present illustration the chart for averages is centered about the normal value of 2.0000, while the chart for ranges is placed with the zero near the bottom of the space for this chart.

c. Inspecting; Computing Averages and Ranges.—At 0830[1] the first sample of 5 bushings was taken from the production line, the diameter of each was measured, to the nearest thousandth of an inch, and the measurements 2.003, 2.001, 1.997, 2.002, and 1.999 were entered under 1 on the form as shown in Table 22. Each entry represents a value of the variable X which appeared in the frequency distribution of Table 16*b*. For the control chart the arithmetic average or mean of X for each sample is needed, and also the range.

To compute the arithmetic average we first add the diameters of the 5 bushings and record this sum, 10.002, as shown for the total. This sum divided by 5, the number of bushings in the sample, gives the arithmetic average 2.0004. This average is designated by $\bar{X}$ (which is read "X bar," or "bar X") and should always be computed and recorded to one decimal more than that to which the measurements were taken. The average has been entered in the table opposite Ave. $\bar{X}$ on the form, and in the tabulation under $\bar{X}$ and opposite 1.

To compute the range the smallest diameter is subtracted from the largest. For the first sample, 1.997 subtracted from 2.003 gives a range R of 0.006, which value is entered in the table opposite range R, on the form, and in the tabulation under R and opposite 1.

At 0855 the second sample of 5 bushings was taken and the same procedure was followed for measuring the diameters, recording the data, computing the average and the range, and entering the values of $\bar{X}$ and R in the tabulation as was done with the first sample. The remaining samples were taken at the times indicated and Table 22 was completed.

d. Plotting Points on Charts.- As soon as the values of $\bar{X}$ and R have been computed for a sample these values are plotted as heavy round dots on the chart form which was prepared as described in *b* above. The successive points on a chart may or may not be connected by straight lines as the person keeping the chart prefers.

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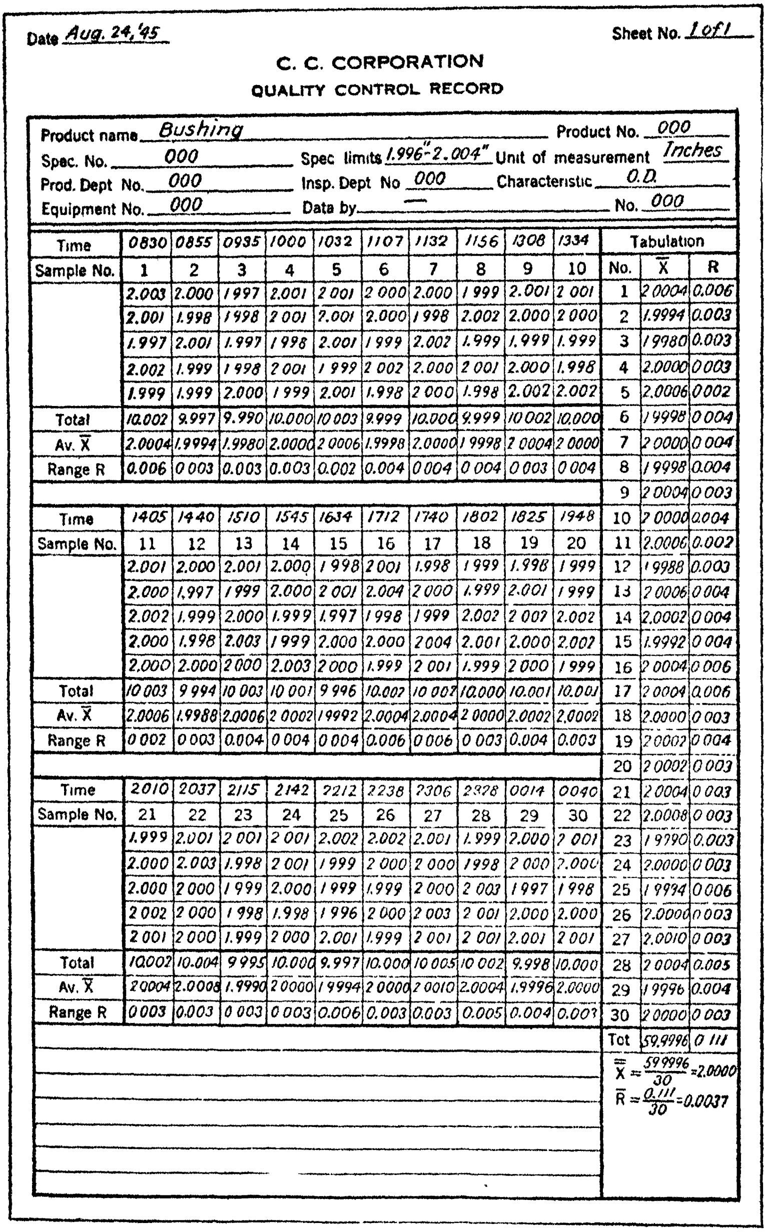
[1] Twenty-four-hour time is indicated in order to avoid confusion between morning and afternoon. This practice was followed so extensively during the Second World War that no trouble should be experienced in reading the time correctly.

TABLE 22.



Date *Aug. 24, '45* Sheet No. *1 of 1*

C. C. CORPORATION

QUALITY CONTROL RECORD

Product name *Bushing* Product No. *000*

Spec. No. *000* Spec limits *1.996"-2.004"* Unit of measurement *Inches*

Prod. Dept No. *000* Insp. Dept No *000* Characteristic *O.D.*

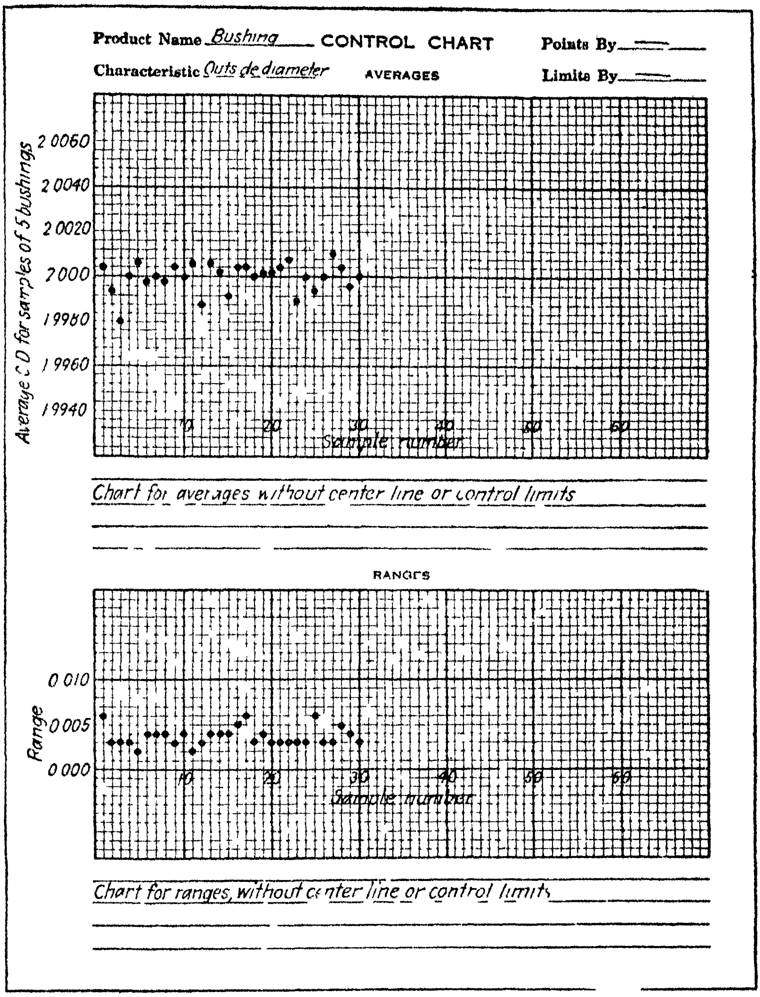
Equipment No. *000* Data by *—* No. *000*

Time	0830	0855	0935	1000	1032	1107	1132	1156	1308	1334	Tabulation		
Sample No.	1	2	3	4	5	6	7	8	9	10	No.	$\bar{X}$	R
	2.003	2.000	1 997	2.001	2 001	2 000	2.000	1 999	2.001	2 001	1	2 0004	0.006
	2.001	1.998	1 998	2 001	2.001	2.000	1 998	2.002	2.000	2 000	2	1.9994	0.003
	1.997	2.001	1.997	1 998	2.001	1 999	2.002	1.999	1.999	1.999	3	1 9980	0.003
	2.002	1.999	1 998	2 001	1 999	2 002	2.000	2 001	2.000	1.998	4	2.0000	0 003
	1.999	1.999	2.000	1 999	2.001	1.998	2 000	1.998	2.002	2.002	5	2.0006	0 002
Total	10.002	9.997	9.990	10.000	10 003	9.999	10.000	9.999	10 002	10.000	6	1 9998	0 004
Av. $\bar{X}$	2.0004	1.9994	1.9980	2.0000	2 0006	1.9998	2.0000	1 9998	2 0004	2 0000	7	2 0000	0 004
Range R	0.006	0 003	0.003	0.003	0.002	0.004	0 004	0 004	0 003	0 004	8	1 9998	0.004
											9	2 0004	0 003
Time	1405	1440	1510	1545	1634	1712	1740	1802	1825	1948	10	2 0000	0.004
Sample No.	11	12	13	14	15	16	17	18	19	20	11	2.0006	0.002
	2.001	2.000	2.001	2.000	1 998	2 001	1.998	1 999	1.998	1 999	12	1 9988	0.003
	2.000	1.997	1 999	2.000	2 001	2.004	2 000	1.999	2.001	1 999	13	2 0006	0 004
	2.002	1.999	2.000	1.999	1.997	1 998	1 999	2.002	2 002	2.002	14	2.0002	0 004
	2.000	1.998	2.003	1 999	2.000	2.000	2 004	2.001	2.000	2.002	15	1.9992	0 004
	2.000	2.000	2 000	2.003	2 000	1.999	2 001	1.999	2 000	1 999	16	2 0004	0 006
Total	10 003	9 994	10 003	10 001	9 996	10.002	10 002	10.000	10.001	10.001	17	2 0004	0.006
Av. $\bar{X}$	2.0006	1.9988	2.0006	2 0002	1 9992	2.0004	2.0004	2 0000	2.0002	2.0002	18	2.0000	0 003
Range R	0 002	0 003	0.004	0 004	0 004	0.006	0 006	0 003	0.004	0.003	19	2 0002	0 004
											20	2 0002	0 003
Time	2010	2037	2115	2142	2212	2238	2306	2328	0014	0040	21	2 0004	0 003
Sample No.	21	22	23	24	25	26	27	28	29	30	22	2.0008	0 003
	1.999	2.001	2 001	2 001	2.002	2.002	2.001	1.999	2.000	2 001	23	1 9990	0.003
	2.000	2.003	1.998	2 001	1 999	2 000	2 000	1 998	2 000	2.000	24	2.0000	0 003
	2.000	2 000	1 999	2.000	1 999	1.999	2 000	2 003	1 997	1 998	25	1 9994	0 006
	2 002	2 000	1 998	1.998	1 996	2 000	2 003	2 001	2.000	2.000	26	2.0000	0 003
	2 001	2 000	1.999	2 000	2.001	1.999	2 001	2 001	2.001	2 001	27	2.0010	0 003
Total	10.002	10.004	9 995	10.000	9.997	10.000	10 005	10 002	9.998	10.000	28	2 0004	0.005
Av. $\bar{X}$	2 0004	2.0008	1.9990	2 0000	1 9994	2 0000	2 0010	2.0004	1.9996	2.0000	29	1 9996	0.004
Range R	0 003	0.003	0 003	0 003	0.006	0.003	0.003	0.005	0.004	0.003	30	2 0000	0 003
											Tot	59.9996	0 111

$$\bar{\bar{X}} = \frac{59\,9996}{30} = 2.0000$$

$$\bar{R} = \frac{0.111}{30} = 0.0037$$

Since on the chart for averages each point represents the average $\bar{X}$ for a sample this chart is also called an "$\bar{X}$-chart." Likewise


Product Name Bushing CONTROL CHART Points By

Characteristic Outside diameter AVERAGES Limits By

Chart for averages without center line or control limits

RANGES

Chart for ranges, without center line or control limits

Fig. 22a

the chart for ranges is also called an "R-chart," because each point represents a value of the range R for a sample.

For the illustration with the bushings the location of each chart and the scales are as shown on Fig. 22a. As soon as $\bar{X}$ and R were computed and entered in the appropriate tabulation columns for a sample the value of $\bar{X}$ and of R were plotted on their respective charts. The charts with all points plotted are shown on Fig. 22*a*.

It would be well for the reader to compare the chart for averages of Fig. 22*a* with the chart for diameters of individual bushings of Fig. 17*b*. The present chart furnishes a better index of the condition of control than was done by the chart of Fig. 17*b*, but it is still inadequate to establish whether or not the process is under control. The condition of control can be determined from either the chart for averages or the chart for ranges if control limits are calculated and corresponding limit lines are drawn on the charts. We shall compute these limits separately for the two charts.

e. Control Limits or Action Limits; General.—Three important lines with every control chart are

1. *Center Line.*—A center line is drawn corresponding to the average value. This average should be very nearly equal to the normal or desired value of the item, and may be equal to it to the accuracy with which measurements are made. The center line is drawn on the chart as a solid line, and may be designated by ℄.

2. *Lower Control Line.*—A line is drawn which represents the lower control limit, or lower action limit. This line is shown on the chart as a dashed or dotted line, and may be designated as *L.C.L.*

3. *Upper Control Line.*—A line is drawn which represents the upper control limit or upper action limit. This line is also shown on the chart as a dashed or dotted line and may be designated as *U.C.L.*

The control limits are also called "action limits," because if a point falls below the lower limit or above the upper limit some action should be taken to discover an assignable cause of trouble with the process.

The position of the center line tells whether or not the control is centered satisfactorily. However, when the danger lines are inserted, and their location with reference to specification limits is discussed, it will be found that these danger lines indicate not only whether the spread is satisfactory but also whether the process is properly centered. If the process is controlled, but is not properly centered, control is said to be at the wrong level, being too high or too low depending upon whether the center line is on the high side or the low side.

The lines for the control limits must be so located that the danger limits determined from them will be properly placed with reference to specification limits. If the points for averages as plotted fall inside the lines drawn to show the control limits then the process is said to be under control. (See Fig. 22*b*.) Furthermore, if the process is under control and the danger limits are properly placed with reference to specification limits (careful consideration will be given to this question in *g* below), then the process is said to be under satisfactory control. It is not enough for the process to be under control, it must be under satisfactory control.

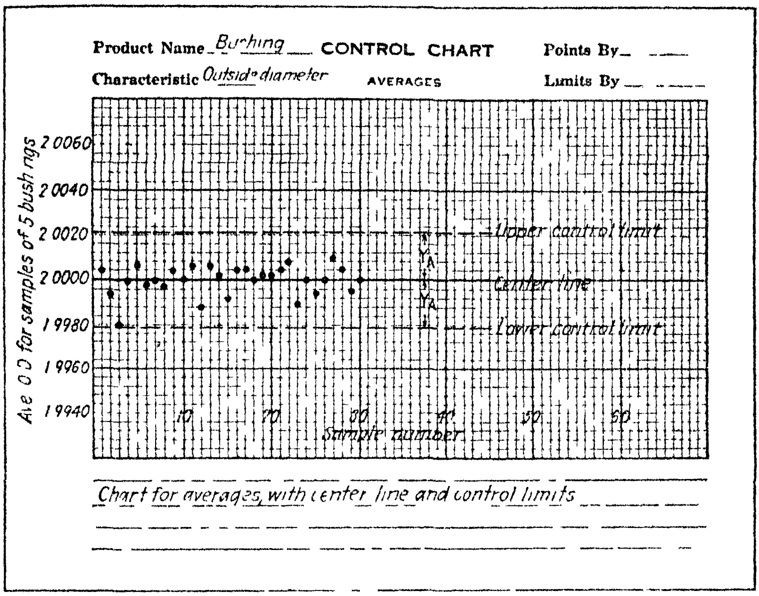


Fig. 22*b*

f. Center Line and Control Limits for Chart for Averages.—In Fig. 22*b* we have redrawn the chart for averages of Fig. 22*a* and have added the center line and the lines for the upper and the lower control limits for our illustration for outside diameters of bushings.

In general to locate the center line it is first necessary to calculate the average of the averages. This is done by finding the sum of the averages for the samples and dividing by the total number of samples. That is, we must add the column in the tabulation headed $\bar{X}$ and divide by the number of entries in the tabulation. This average is designated by $\bar{\bar{X}}$ (read "X double bar" or "double bar X"). It is actually the arithmetic average of all the items.

For the bushings we see, from Table 22, that the sum of the $\bar{X}$ column is 59.9996, and this divided by 30 gives $\bar{\bar{X}}$ as 1.999987, or, to four decimals

$$\bar{\bar{X}} = 2.0000 \tag{1}$$

Hence the center line was drawn on Fig. 22*b* to coincide with the normal outside diameter as given in the specifications.

To determine the control limits we shall now introduce the formulas that are customarily used and that are based on the average range $\bar{R}$.

The location of the lines for the control limits on a chart for averages are given by

$$\bar{\bar{X}} + Y_A \tag{2}$$

for the upper control limit

$$\bar{\bar{X}} - Y_A \tag{3}$$

for the lower control limit. In (2) and (3) Y_A denotes the spread on either side of the center line and its value is found by

$$Y_A = A_2\bar{R} \tag{4}$$

Formulas (2) and (3) are usually combined and written with $A_2\bar{R}$ from (4) in place of Y. This gives

$$\bar{\bar{X}} \pm A_2\bar{R} \tag{5}$$

as the formulas for locating the control limits.

It is seen from (2), (3), and (5) that the control limits are at equal distances from the center line. This is always true for a chart for averages and is true in general for any control chart excepting when a negative result is obtained by subtracting the spread from the average, and then the lower limit is taken at zero. This is the case in a chart for ranges if the sample size is 6 or less, and in Chap. IV it will be found to be the case for some of the charts where inspection is by attributes.

The A_2 appearing in (4) is a factor that depends upon the sample size, while the $\bar{R}$ is the average of the ranges. Values of A_2 for samples of 2 to 10 items are given in Table III, page 158. In the illustrations and the exercises of this chapter, we shall use only a sample size of 4 or 5 so we shall now give the values of A_2 corresponding to these sample sizes

$$(6)\qquad \begin{cases} A_2 = 0.729 & \text{for a sample of 4 items} \\ A_2 = 0.577 & \text{for a sample of 5 items} \end{cases}$$

Since in our illustration there are 5 bushings in each sample the value of A_2 as obtained from (6) is 0.577. The average $\bar{R}$ of the ranges is found by adding the column under R in the tabulation of Table 22 and dividing by 30. From that table this is seen to be

$$(7)\qquad \bar{R} = 0.0037$$

We can now determine Y_A from (4), (6), and (7) and find

$$(8)\qquad Y_A = A_2\bar{R} = 0.577 \times 0.0037 = 0.0021$$

to four decimals.

Now from (1), (2), (3), (4), and (8) we have

$$(9)\qquad \begin{cases} \bar{\bar{X}} + Y_A = \bar{\bar{X}} + A_2\bar{R} = 2.0000 + 0.0021 \\ \qquad = 2.0021 \quad \text{for the upper control limit} \\ \bar{\bar{X}} - Y_A = \bar{\bar{X}} - A_2\bar{R} = 2.0000 - 0.0021 \\ \qquad = 1.9979 \quad \text{for the lower control limit} \end{cases}$$

By extending the combined formulas as given in (5) the formulas in (9) may be written as

$$(10)\qquad \bar{\bar{X}} \pm A_2\bar{R} = 2.0000 \pm 0.0021 = \begin{cases} 2.0021 & \text{U.C.L.} \\ 1.9979 & \text{L.C.L.} \end{cases}$$

Lines at the locations given in (9) or (10) have been drawn and identified on Fig. 22*b*. Since the point for each sample falls within the control limits, the process is under control, and since the center line is actually along the desired normal value, the control is at a satisfactory level. In section (*g*) we shall compute and locate the danger limits for this control chart and see whether the control is satisfactory.

The first point for the third sample is close to the lower control limit, but as it is inside of that limit no lack of control is shown. However, the downward trend of the first three samples might indicate the beginning of trouble somewhere in the process, and consequently the fourth sample should be observed with extra care. Since the fourth and several immediately following samples have quite normal averages the trend indicated nothing but a chance variation. The generally upward trend from the fifteenth to the twenty-second samples would also be noted and the next few samples should be watched to see if perhaps a trend were starting that could mean some change was taking place in the process. In this case also there was no cause for worry.

g. Danger Limits for Chart for Averages.—In Art. 17 it was stated that substitute formulas would be introduced in place of those used in courses on statistical methods for calculating the danger limits. The substitute formulas for the locations of the danger limits on control charts for averages are

$$X + Z \tag{11}$$

for the upper danger limit, and

$$\bar{\bar{X}} - Z \tag{12}$$

for the lower danger limit, where

$$Z = \sqrt{n}Y_A = \sqrt{n}A_2R \tag{13}$$

and n is the sample size.

For the illustration with bushings n is 5 and $\sqrt{n} = \sqrt{5} = 2.236$. Substituting this and the value of Y_A from (8) in (13) we find

$$Z = \sqrt{n}Y_A = \sqrt{5}(0.0021) = (2.236)(0.0021) = 0.0047 \tag{14}$$

to four decimals. Substituting in (11) and (12) the value of $\bar{\bar{X}}$ in (1), and of Z in (14), we have

$$\tag{15} \left\{ \begin{aligned} \bar{\bar{X}} + Z &= \bar{\bar{X}} + \sqrt{n}A_2\bar{R} = 2.0000 + 0.0047 \\ &= 2.0047 \quad \text{for upper danger limit} \\ \bar{\bar{X}} - Z &= \bar{\bar{X}} - \sqrt{n}A_2\bar{R} = 2.0000 - 0.0047 \\ &= 1.9953 \quad \text{for lower danger limit} \end{aligned} \right.$$

These formulas in (15) are also frequently combined, as formulas for control limits were combined in (10), and they are then written as

$$(16)\quad X \pm \sqrt{n}A_2\bar{R} = 2.0000 \pm 0.0047 = \begin{cases} 2.0047 & \text{U.D.L.} \\ 1.9953 & \text{L.D.L.} \end{cases}$$

These danger limits are for the same frequency distribution that was discussed in Art. 17. In that article the limits were given as 2.0045 and 1.9955, and the differences between these and the corresponding ones of formulas (15) or (16) are negligible. (The limits given in Art. 17 were obtained by use of formulas of statistical methods but the formulas and calculations were not shown.)

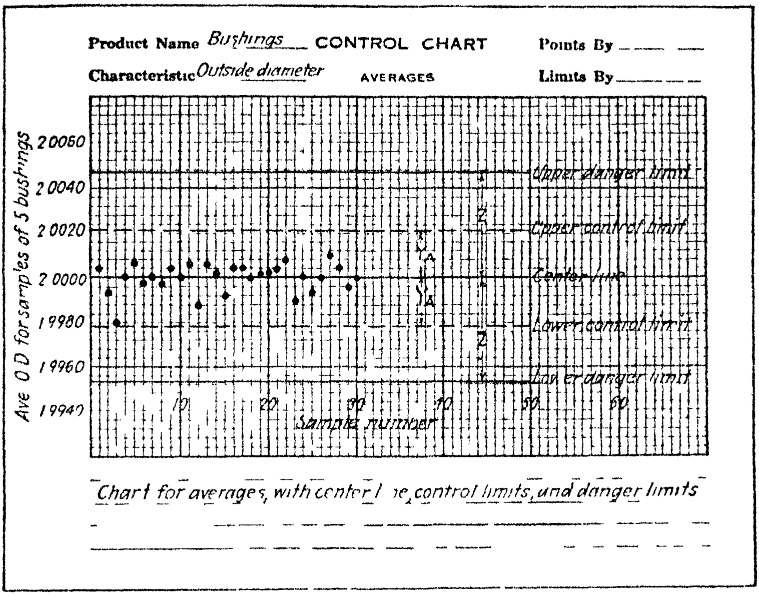


Fig 22c

In Fig. 22c we have redrawn Fig 22b and have added and identified the lines to show the upper and the lower danger limits. It should be noted that the control limits and the danger limits are set by the actual results of the production process and not by the specifications. Since a condition of control is indicated for the bushings, by the fact that all points of Fig. 22b or Fig. 22c fall within control limits, there is only a very slight chance (less than 3 in 1,000) that any point plotted, for an average outside diameter for a sample of 5 bushings, will fall outside either of the control limits. If one of these points should fall outside of a control limit there is grave danger that one individual bushing (or more than one) would have a diameter outside of a danger limit, and hence outside of specifications. If a point falls outside of a control limit, it should not be assumed that it was due to chance but a search should be made to see if something has gone wrong with the process for which an assignable cause can be found.

It is suggested that the reader try to determine certain features from the tabulation of averages of Table 22, and then that he use Fig. 22*b*, or Fig. 22*c*, to determine the same features. For example, suppose you try to pick the largest and the smallest values of $\bar{X}$ from the tabulation and see how many times it is necessary to run your eyes up and down the column of figures. At a glance the maximum and the minimum values of $\bar{X}$ may be identified from either figure by picking the highest and the lowest points on the chart. After picking the highest and the lowest points on the chart you can be certain that you have selected maximum and minimum values of $\bar{X}$, while if you have attempted to select these from a tabulation you would probably be uncertain whether or not you may have overlooked some entries. Suppose next that you try to determine the frequency with which a particular average, say 2.0004, appears. You would have to read the column of figures in the tabulation column for $\bar{X}$ very slowly and would still wonder whether or not you recorded every one. The frequency can be found readily from the chart by choosing the appropriate horizontal line and counting the number of dots on it. A few such comparisons should show how useful the charts are for identifying certain features.

If we should make a frequency distribution and a frequency curve for the data entered under $\bar{X}$ of the tabulation in Table 22, the danger limits for this curve would be very approximately equal to the control limits of Figs. 22*b* and 22*c*. Such a curve would be known as a "frequency curve for averages."

The danger limits for a process that has a normal and approximately symmetrical frequency distribution, whether computed by formulas (15) or (16) or by formulas of statistical methods, will fall outside of the largest and the smallest values that are acceptable under specifications. That is, if the largest and the smallest values given in the specifications are acceptable values, and if items with such values are included in a few samples, then the danger limits when calculated will be outside of the specification limits. For satisfactory control, however, the calculated value of each danger limit must not be outside the corresponding specification limit by as much as the fraction of a unit to which the items are measured. Let us illustrate this statement by means of the limits for the bushings.

The outside diameters of the bushings were measured to the nearest thousandth of an inch, with specifications calling for 2 inches ± 0.004 inch. Diameters of 1.996 inches and 2.004 inches were acceptable and both were found among the samples as measured and recorded in the tables. From the statement in the immediately preceding paragraph the lower danger limit should be between 1.9950 inches and 1.9960 inches, and the upper limit between 2.0040 inches and 2.0050 inches. The limits of 1.9953 inches and 2.0047 inches, as given in (16), are seen to satisfy these conditions.

In an application in a factory so much care is taken to control the quality of an item that quite frequently a process becomes better than is necessary, and then the danger limits may fall inside the limits as set by the specifications. For example, suppose that, with our illustration with the bushings, no bushings were made with diameters as small as 1.996 inches or as large as 2.004 inches (the smallest and the largest allowed under specifications), that there were very few with diameters of 1.997 inches and 2.003 inches, and that there was a greater number with diameters of 2.000 inches, or very nearly 2.000 inches, than was indicated by the theoretical distribution of Table 16*c*. In this case $\bar{R}$, and consequently Z, would be small enough for the lower danger limit to be greater than 1.996 inches and the upper one to be less than 2.004 inches.

Very careful thought should be given to the discussion in the last three paragraphs above, regarding the location of the danger limits in reference to the specification limits. Many errors arise in applications of control charts in practice by erroneously assuming that limits which correspond to the ones that we have called danger limits should fall inside of specification limits when those specification limits represent acceptable values. In those factories where the control of quality is carefully watched by all departments it is not uncommon for the production process to become so perfected that items produced are better than required by specifications. In such cases items with extreme values that may be allowed by specifications are rarely made, and hence the danger limits that are fixed by the process usually fall inside of specification limits. However, if extreme values are made in the proportion permitted by a standard frequency distribution, then the danger limits will fall outside of specification limits, and, if they are not outside by more than the fraction of a unit to which the items are measured, then the process should be considered as under a satisfactory condition of control.

h. Center Line and Control Limits for Chart for Ranges.—The center line on a chart for ranges is located by $\bar{R}$, the average value of the range. The control limits are spaced equally above and below the center line if the sample size is 7, or more. The lower limit is taken as zero if the sample size is 6 or less, and in these cases the center line is not centered between the control limits.

The location of the upper control limit on an R-chart is given by the formula

$$\bar{R} + Y_R \tag{17}$$

while the location of the lower control limit is

$$\begin{cases} \bar{R} - Y_R & \text{if the sample size is 7 or more} \\ 0 & \text{if the sample size is 6 or less} \end{cases} \tag{18}$$

The value of Y_R to be used in (17) and (18) is obtained from

$$Y_R = D\bar{R} \tag{19}$$

where D is a factor whose value depends upon the sample size. Values of D for various sample sizes may be found in Table III. The values of D are recorded in (20) for samples of 4 or 5, since we are limiting our applications in this chapter to samples with those sizes.

$$\begin{cases} D = 1.282 & \text{for a sample of 4} \\ D = 1.114 & \text{for a sample of 5} \end{cases} \tag{20}$$

We may replace Y_R by $D\bar{R}$ from (19) and then combine (17) and (18) and write the formulas for locating the control limits as

$$\bar{R} \pm D\bar{R} \tag{21}$$

If we use (21) we must remember that the lower limit is to be taken as zero when $\bar{R} - D\bar{R}$ is negative. This will be the case when D is greater than 1, as it is if the sample size is 6 or less.

Substituting in (19) the value of $\bar{R} = 0.0037$, from (7), and of $D = 1.114$, from (20), for a sample of 5 we find, for the bushings,

$$Y_R = D\bar{R} = 1.114 \times 0.0037 = 0.0041 \tag{22}$$

to four decimals. Using the formulas in (21) and the value of $\bar{R}$ in (7) and of $D\bar{R}$ in (22) we may write

$$\bar{R} \pm D\bar{R} = 0.0037 \pm 0.0041 = \begin{cases} 0.0078 & \text{U.C.L.} \\ 0.0000 & \text{L.C.L.}^1 \end{cases} \tag{23}$$

for the locations of the control limits.

The chart for ranges of Fig. 22*a* has been redrawn on Fig. 22*d*, and the center lines and the control lines have been located and identified.

A chart for ranges is not in general as positive an index of the condition of control as a chart for averages. If points on a chart for ranges fall outside of control limits there is an indication of lack of control, but all points may fall inside control limits even though the process may not be under control. There are, however, some cases in which a lack of control may be revealed more quickly on a chart for ranges than on a chart for averages.

The two charts are usually used together during the preliminary stages of determining whether or not a process is under control, and a search is made to see if something has gone wrong with the process when a point on either chart falls outside of a control limit. If in an application one chart is found to be a sensitive index of the control of quality while the other one gives little or

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[1] This is not a true equality for the lower control limit since 0.0037 − 0.0041 is −0.0004 and not zero. However, we shall use this condensed form of indicating the control limits, with zero replacing any negative results obtained in computing the location of the lower control limit. This will conform to the lower control limit as given in (18).

no indication of the condition of control, then the more sensitive one is continued and the other one is dropped.

In the next section we will show the two charts drawn together and will give further attention to the question of getting a process under control.

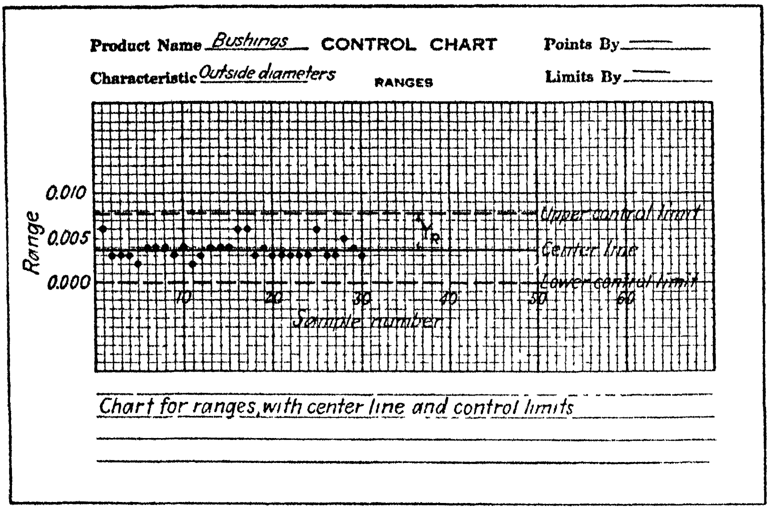


FIG. 22*d*.

i. Preliminary Chart.—We have drawn the chart for averages and the chart for ranges lined up together on the same sheet on Fig. 22*e*, with center lines, control limits, and danger limits located and identified.

At the stage represented by the charts of Fig. 22*e*, the charts are really preliminary charts and can be used to obtain a preliminary analysis of the state of control. If satisfactory control has been established then the average value and the control limits can be adopted as standard values, and the center line and the control lines of the preliminary charts can be projected forward and the charts used to control quality during production. However, if the process is not in control then the quality control experts will show the engineers the evidence of trouble and the engineers will search out the causes of trouble and will take steps to make necessary corrections or adjustments. The engineers will continue to search for the sources of trouble and to apply corrective measures until the process is brought under satisfactory control. When this condition has been attained then new averages and limit values, based on 25 or 30 samples taken

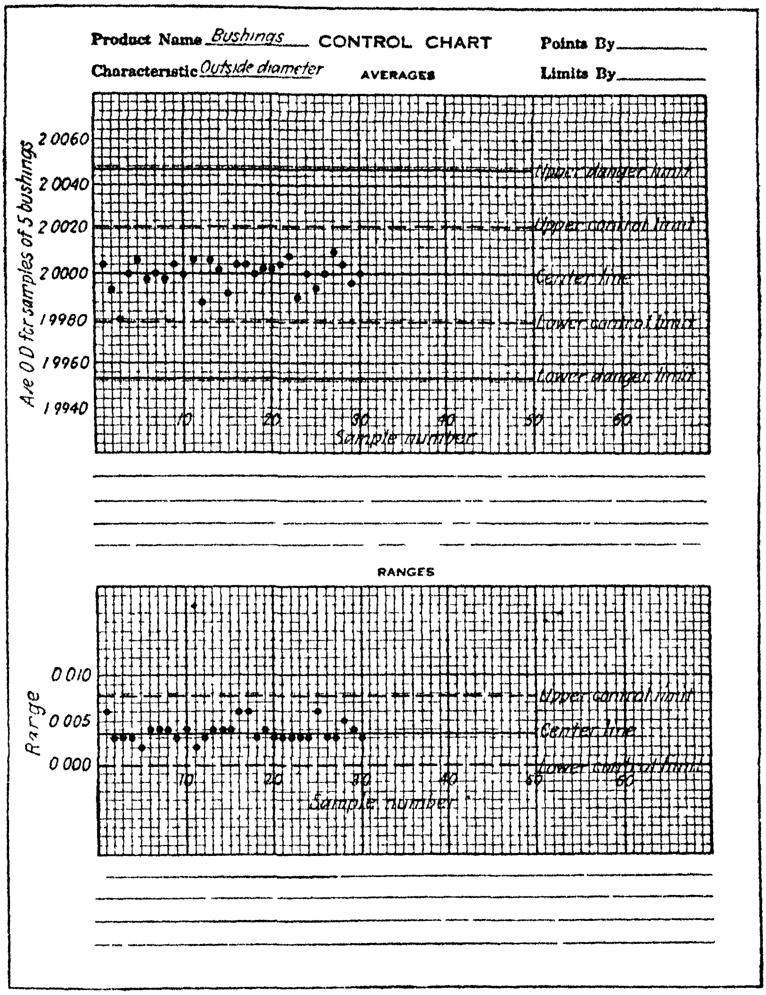


FIG. 22e.

after satisfactory control has been attained, will be calculated and new limit lines will be placed on the charts. These new values and limits will be adopted as standards and the lines will be projected forward and used as guides for determining the condition of quality of future production. If the quality remains satisfactory as production continues, the plotted points will remain between these control limits. As soon as the condition of control becomes satisfactory and remains stable the frequency of taking samples can be reduced. Then it is necessary to take samples only often enough to see that the condition of control remains satisfactory.

The preliminary averages and limits as considered above are computed after production has continued long enough for 30 samples to be inspected. The time elapsed before 30 samples (or even 25) can be measured could represent a serious delay in determining the condition of control for the process. There are two methods by which tentative preliminary control values may be obtained at the time we start the control chart. In either case, however, the tentative values should be used only until 25 or 30 samples have been measured, and then new averages and limits, that would be dependent upon the current process, should be computed.

First Method. It may be that a control chart is to be started for a process that has been in operation for some time and that, from past performance, data are available from which to calculate limits. If production is to continue without any essential change, and the data from past performance can be accepted as representative of current conditions, then the average and limits based on these data can be used as tentative control values.

Second Method. If data are not available from former productions then, for some applications, tentative control limits for a chart for averages can be computed on the basis of specifications. To do this we should locate the center line by taking $\bar{X}$ as equal to the normal or the desired value as given in specifications. The distance Z from the center line to either danger limit should be selected to satisfy the conditions discussed at the end of section *g*. Thus Z should be greater than the distance from the center line to a specification limit, but should not exceed this by an amount equal to the smallest fraction of a unit to which measurements are recorded. This value of Z divided by $\sqrt{n}$ (where n is the sample size) will give a tentative value of Y_A, the distance from the center line to either control limit.

In those cases where tentative limits are found by the second method we should be particularly careful to calculate the control limits, based on the actual process, as soon as 25 or 30 samples have been measured. If the plotted points fall inside these control limits as determined from the process, then the process will be under control, and if these limits are approximately equal to, or are inside, those as set above from specifications, then the control will be satisfactory.

Let us apply this discussion to the illustration with bushings. Since the outside diameters of the bushings were so definitely controlled, and had a frequency distribution that was so nearly that of the normal theoretical distribution, the limits determined from the process should closely approximate those determined from specifications.

As specifications called for a diameter of 2 inches ± 0.004 inch and measurements were to the nearest thousandth of an inch, let us take $\bar{\bar{X}}$ as 2.0000 and Z as $0.0040 + 0.0009$, or $Z = 0.0049$. The danger limits are thus $2.0000 + 0.0049 = 2.0049$ for the upper limit and $2.0000 - 0.0049 = 1.9951$ for the lower limit. Then we have

$$Y_A = \frac{Z}{\sqrt{n}} = \frac{0.0049}{\sqrt{5}} = \frac{0.0049}{2.236} = 0.0022$$

and the tentative, preliminary control limits are

$$(24) \qquad \bar{\bar{X}} \pm Y_A = 2.0000 \pm 0.0022 = \begin{cases} 2.0022 & \text{U.C.L.} \\ 1.9978 & \text{L.C.L.} \end{cases}$$

Comparing these with the control limits of 2.0021 and 1.9979, found in (9) or (10), we see that tentative control limits determined from specifications for outside diameters of the bushings give satisfactory tentative preliminary control limits.

23. Process Control

We shall now make a control chart analysis that will demonstrate the use of a chart in controlling quality during production. To do this we shall use an illustration with data obtained from a plant that was engaged in war work. Slight adjustments have been made with the data in order to complete the story within the boundaries of Fig. 23.

The product selected for the illustration was a small gear, and the characteristic selected was the outside diameter of the gear. Specifications called for 1.996 inches/1.990 inches, or 1.993 inches ± 0.003 inch, and both limits of 1.996 inches and 1.990 inches were acceptable values. Samples of 4 were taken from the production line at approximately half-hour intervals and the outside diameter of each was measured to the nearest thousandth of an inch.

Since measurements were made to the nearest thousandth of an inch, the average in each sample should be given to the nearest ten thousandth, and we must decide what to do if the fifth digit is a 5, as it will be sometimes for a sample size of 4. Of course we could use the rule of always increasing to the next size, as we did with our illustration with heights of students. However, a better rule here is to make the resulting average an even number. Thus an average of 1.99245 would be written 1.9924, while one of 1.99255 would be written 1.9926.

No past data were available from which to obtain preliminary control limits, so we must obtain preliminary limits from the process after measurements from 25 to 30 samples of 4 are available. However, we can follow the second method of Art. 22*i* and can find tentative limits which will give a fair index of the condition of control during the early stages of collecting the data.

For tentative danger limits, we start with $\bar{X} = 1.9930$ and choose $Z = 0.0030 + 0.0009 = 0.0039$, so that the upper and the lower danger limits, if shown, would be located at 1.9969 and 1.9891, respectively. Since these will have no special use on our chart we will not show them. The tentative control limits will be useful and will be shown as dotted lines for the first 30 points on the chart. To locate them we use $Z = 0.0039$ and $n = 4$ and find

$$(1)\qquad Y_A = \frac{Z}{\sqrt{n}} = \frac{0.0039}{2} = 0.00195 = 0.0020$$

Hence the locations of the tentative control limits are

$$(2)\qquad \bar{\bar{X}} \pm Y_A = 1.9930 \pm 0.0020 = \begin{cases} 1.9950 & \text{U.C.L.} \\ 1.9910 & \text{L.C.L.} \end{cases}$$

We now lay out the forms for charts for averages and ranges, with scales as shown in Fig. 23. On the chart for averages, the tentative control lines are shown dotted for the first 30 samples.

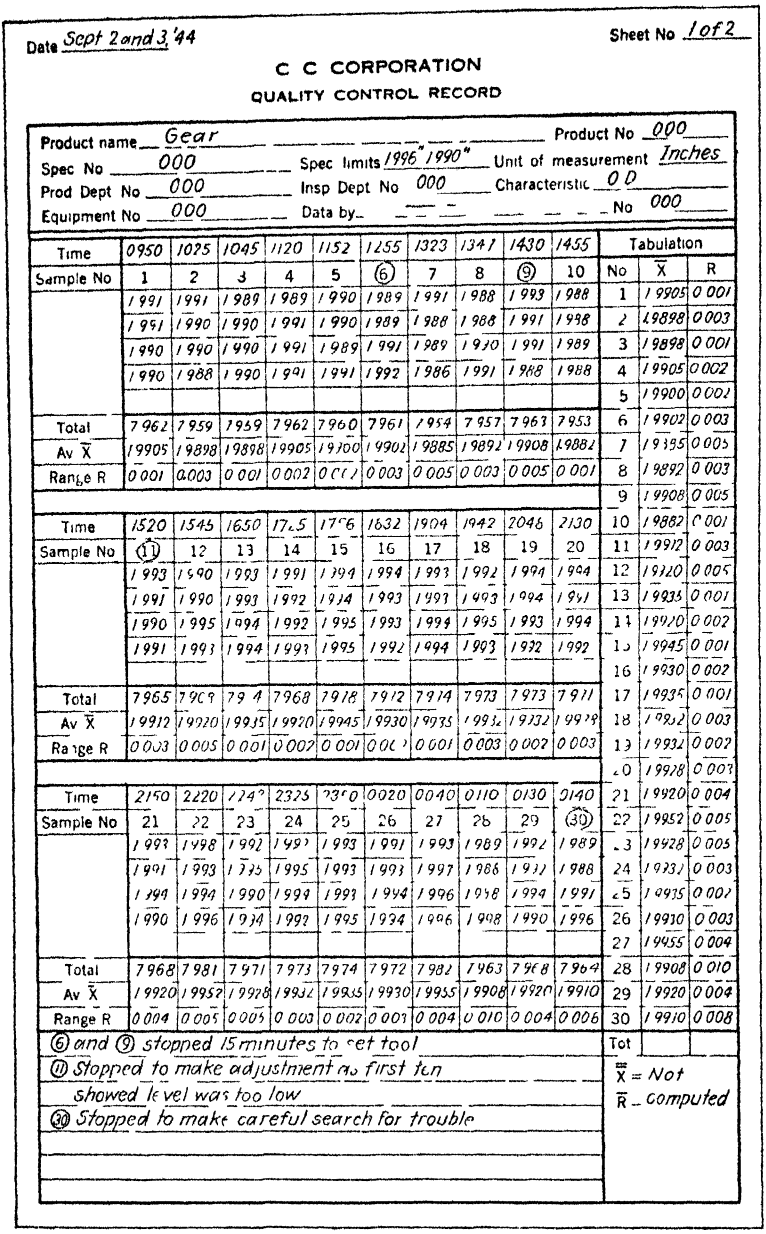
Beginning at 0950 samples of 4 were taken from the production line, the average and the range for each sample were computed, recorded in the tabulation columns of Table 23, and the corresponding points were plotted on Fig. 23. As each point on the chart for averages fell below the lower tentative control limit, it soon became evident that the process level was too low. An investigation revealed that the operator was purposely running low in order to reduce the amount of material that would have to be removed by the grinding operation. After the tenth sample the machine was stopped and proper adjustments were made. Also the operator and supervisor were reinstructed on the proper method of setting tools so that trouble, immediately after setting a tool, such as is indicated following samples 6 and 9, could be avoided.

When production was resumed after the adjustment, the eleventh point was low, being at the lower tentative control limit, but from sample 12 to 21 the points were quite satisfactorily located. Number 15 was a little high but was still inside the upper tentative control limit. When sample 22 appeared above the upper tentative control limit a search was made for the cause of trouble. However, production was continued until the high point at 27, the low points at 28 and 30, and the wide range for the twenty-eighth and the thirtieth samples showed definite evidence of trouble.

The machine was shut down and left until the maintenance men could make a thorough check. They found the chief cause for trouble was a worn bearing. This was replaced with a new one, some minor adjustments were made, and the machine was put back in operation. The inspection record and computations following repairs appear on the second sheet of Table 23, and the corresponding points are shown from 31 to 60 on the chart for averages and the chart for ranges of Fig. 23.

Since the points from 31 to 60 are satisfactorily inside the tentative limits we shall now compute control limits that are fixed by the process from which these points were obtained. Then from these control limits we can determine whether or not

TABLE 23



Date Scpt 2 and 3, '44

Sheet No 1 of 2

C C CORPORATION

QUALITY CONTROL RECORD

Product name Gear

Product No 000

Spec No 000

Spec limits 1996" 1990"

Unit of measurement Inches

Prod Dept No 000

Insp Dept No 000

Characteristic O D

Equipment No 000

Data by

No 000

Time	0950	1025	1045	1120	1152	1255	1323	1341	1430	1455
Sample No	1	2	3	4	5	⑥	7	8	⑨	10
	1 991	1 991	1 989	1 989	1 990	1 989	1 991	1 988	1 993	1 988
	1 991	1 990	1 990	1 991	1 990	1 989	1 988	1 988	1 991	1 988
	1 990	1 990	1 990	1 991	1 989	1 991	1 989	1 990	1 991	1 989
	1 990	1 988	1 990	1 991	1 991	1 992	1 986	1 991	1 988	1 988
Total	7 962	7 959	7 959	7 962	7 960	7 961	7 954	7 957	7 963	7 953
Av $\bar{X}$	1 9905	1 9898	1 9898	1 9905	1 9900	1 9902	1 9885	1 9892	1 9908	1 9882
Range R	0 001	0.003	0 001	0 002	0 002	0 003	0 005	0 003	0 005	0 001

Time	1520	1545	1650	1725	1756	1632	1904	1942	2046	2130
Sample No	⑪	12	13	14	15	16	17	18	19	20
	1 993	1 990	1 993	1 991	1 994	1 994	1 993	1 992	1 994	1 994
	1 991	1 990	1 993	1 992	1 994	1 993	1 993	1 993	1 994	1 991
	1 990	1 995	1 994	1 992	1 995	1 993	1 994	1 995	1 993	1 994
	1 991	1 993	1 994	1 993	1 995	1 992	1 994	1 993	1 992	1 992
Total	7 965	7 969	7 974	7 968	7 978	7 972	7 974	7 973	7 973	7 971
Av $\bar{X}$	1 9912	1 9920	1 9935	1 9920	1 9945	1 9930	1 9935	1 9932	1 9932	1 9928
Range R	0 003	0 005	0 001	0 002	0 001	0 002	0 001	0 003	0 002	0 003

Time	2150	2220	2240	2325	2350	0020	0040	0110	0130	0140
Sample No	21	22	23	24	25	26	27	28	29	㉚
	1 993	1 998	1 992	1 997	1 993	1 991	1 993	1 989	1 992	1 989
	1 991	1 993	1 995	1 995	1 993	1 993	1 997	1 988	1 992	1 988
	1 994	1 994	1 990	1 994	1 993	1 994	1 996	1 998	1 994	1 991
	1 990	1 996	1 994	1 992	1 995	1 994	1 996	1 998	1 990	1 996
Total	7 968	7 981	7 971	7 973	7 974	7 972	7 982	7 963	7 968	7 964
Av $\bar{X}$	1 9920	1 9952	1 9928	1 9932	1 9935	1 9930	1 9955	1 9908	1 9920	1 9910
Range R	0 004	0 005	0 005	0 003	0 002	0 003	0 004	0 010	0 004	0 006

⑥ and ⑨ stopped 15 minutes to set tool

⑪ Stopped to make adjustment as first ten showed level was too low

㉚ Stopped to make careful search for trouble

Tabulation		
No	$\bar{X}$	R
1	1 9905	0 001
2	1 9898	0 003
3	1 9898	0 001
4	1 9905	0 002
5	1 9900	0 002
6	1 9902	0 003
7	1 9885	0 005
8	1 9892	0 003
9	1 9908	0 005
10	1 9882	0 001
11	1 9912	0 003
12	1 9920	0 005
13	1 9935	0 001
14	1 9920	0 002
15	1 9945	0 001
16	1 9930	0 002
17	1 9935	0 001
18	1 9932	0 003
19	1 9932	0 002
20	1 9928	0 003
21	1 9920	0 004
22	1 9952	0 005
23	1 9928	0 005
24	1 9932	0 003
25	1 9935	0 002
26	1 9930	0 003
27	1 9955	0 004
28	1 9908	0 010
29	1 9920	0 004
30	1 9910	0 008
Tot		

$\bar{\bar{X}}$ = Not

$\bar{R}$ – computed

Table 23 (*continued*)

Date Sept 2-3,'44

Sheet No. 2 of 2

C. C. CORPORATION

QUALITY CONTROL RECORD

Product name Gear

Product No. 000

Spec No. 000

Spec. limits 1.996"-1 990"

Unit of measurement Inches

Prod Dept No 000

Insp Dept No 000

Characteristic O.D.

Equipment No 000

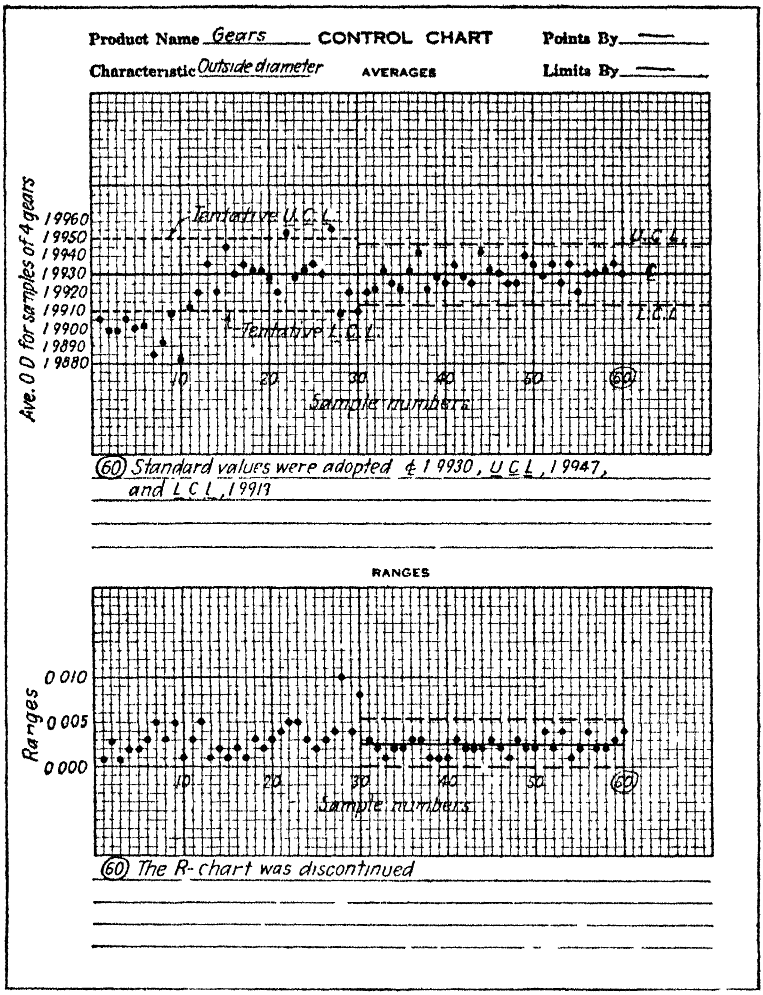
Data by ——

No 000

Time	1130	1250	1322	1354	1435	1454	1534	1655	1720	1747	Tabulation		
Sample No	1	2	3	4	5	6	7	8	9	10	No	$\bar{X}$	R
	1 992	1.992	1.993	1 991	1 992	1 992	1 993	1 992	1 992	1.992	1	1.9920	0.003
	1 994	1 993	1.993	1 993	1 991	1 993	1 994	1 993	1 993	1 993	2	1 9922	0.002
	1.991	1.993	1.993	1 993	1.993	1 993	1 994	1 992	1 993	1 993	3	1 9932	0 001
	1.991	1 991	1 994	1 993	1 993	1 995	1 996	1 992	1 993	1 992	4	1 9925	0.002
											5	1.9922	0.002
Total	7.968	7 969	7 973	7 970	7 969	7 973	7 971	7969	7 971	7970	6	1.9932	0.003
Av $\bar{X}$	1.9920	1 9922	1 9932	1 9925	1 9922	1 9932	1 9942	1 9922	1 9928	1.9925	7	1.9942	0 003
Range R	0 003	0 002	0 001	0.002	0.002	0 003	0 003	0.001	0 001	0 001	8	1 9922	0.001
											9	1 9928	0.001
Time	1835	1858	1920	1950	2048	2130	2205	2240	2315	2352	10	1 9925	0 001
Sample No	11	12	13	14	15	16	17	18	19	20	11	1.9935	0 003
	1.993	1 994	1 991	1 995	1 992	1 994	1 992	1 991	1 994	1 994	12	1 9928	0.002
	1 992	1 993	1 993	1 993	1 994	1 992	1 993	1 993	1.993	1 994	13	1 9925	0 002
	1 995	1 992	1 993	1 995	1 995	1 992	1 992	1 994	1 994	1 994	14	1 9942	0.002
	1 994	1 992	1 993	1 994	1 992	1 994	1 993	1 992	1 995	1.992	15	1 9932	0 003
											16	1 9930	0 002
Total	7 974	7 971	7 970	7 977	7 973	7.972	7 970	7 970	7 976	7.974	17	1 9925	0 001
Av $\bar{X}$	1 9935	1 9928	1.9925	1 9942	1 9932	1 9930	1 9925	1 9925	1 9940	1 9935	18	1.9925	0 003
Range R	0 003	0 002	0 002	0 002	0 003	0 002	0 001	0.003	0.002	0 002	19	1 9940	0.002
											20	1 9935	0 002
Time	0015	0050	0140	0205	0245	0320	0355	0450	0525	0610	21	1.9928	0.004
Sample No	21	22	23	24	25	26	27	28	29	30	22	1.9935	0.002
	1.992	1 993	1 990	1 993	1 991	1 991	1 992	1 992	1 992	1 991	23	1 9925	0.004
	1 993	1 993	1 994	1 993	1 991	1 993	1 992	1 993	1 993	1 993	24	1 9935	0 001
	1 995	1 993	1 993	1.994	1 993	1.993	1 994	1 994	1 994	1.993	25	1 9920	0.002
	1.991	1 995	1 993	1 994	1 993	1 995	1 994	1 994	1 995	1 995	26	1 9930	0.004
											27	1 9930	0.002
Total	7 971	7 974	7 970	7 974	7 968	7 972	7 972	7 973	7 974	7 972	28	1.9932	0 002
Av $\bar{X}$	1 9928	1 9935	1 9925	1.9935	1 9920	1 9930	1 9930	1 9932	1 9935	1 9930	29	1 9935	0.003
Range R	0 004	0 002	0.004	0 001	0 002	0 004	0 002	0.002	0 003	0.004	30	1 9930	0.004
											Tot.	59 7887	0 069

$$\bar{\bar{X}} = \frac{59.7887}{30} = 1.99296$$

$$\bar{R} = \frac{0.069}{30} = 0.0023$$



(60) Standard values were adopted ¢ 1 9930, U C L, 1 9947, and L C L, 1 9913

(60) The R-chart was discontinued

FIG. 23

the process is under control and, if so, whether the control is satisfactory.

Using the data of the second sheet of Table 23, we shall first compute limits for the chart for averages. From the computations on that sheet we have

$$\bar{\bar{X}} = 1.9930 \tag{3}$$

to four decimals, and

$$\bar{R} = 0.0023 \tag{4}$$

To locate the control limits we must compute the distance Y_A from the center line $\bar{\bar{X}}$ to either of the control limits. As the sample size is 4 we substitute

$$A_2 = 0.729 \tag{5}$$

given in (5), Art. 22 (or in Table III), and the value of $\bar{R}$ given in (3) above, in (4), Art. 22, and obtain

$$Y_A = A_2\bar{R} = 0.729 \times 0.0023 = 0.0017 \tag{6}$$

to four decimals. Now using the value of $\bar{\bar{X}}$ from (3), and Y_A from (6), we may give the locations of the control limits in the condensed form

$$\bar{\bar{X}} \pm A_2\bar{R} = 1.9930 \pm 0.0017 = \begin{cases} 1.9947 & \text{U.C.L.} \\ 1.9913 & \text{L.C.L.} \end{cases} \tag{7}$$

Dashed lines have been drawn for upper and lower control limits from 30 to 60 on the chart of Fig. 23, and these lines have been projected beyond 60 to be used with future production.

Since the plotted points from 30 to 60 of the chart are all inside the control limits, the process is under control, and since the control limits are inside the tentative limits set by specifications the control is satisfactory.

The fact that the control is satisfactory could also be shown by locating the danger limits and noting that they are not more than 0.001 inch beyond the specification limits. To locate these on the chart for averages we use formula (13), Art. 22, with $n = 4$, to obtain the distance Z from the center line to either danger limit. Hence we find

$$Z = Y_A\sqrt{n} = 0.0017 \times 2 = 0.0034 \tag{8}$$

We now use the value of $\bar{\bar{X}}$ in (3) and of Z in (8) and find the locations of the danger limits in the condensed form as

$$\bar{\bar{X}} \pm Z = \bar{\bar{X}} \pm \sqrt{n}A_2\bar{R} = 1.9930 \pm 0.0034$$

$$= \begin{cases} 1.9964 & \text{U.D.L.} \\ 1.9896 & \text{L.D.L.} \end{cases} \tag{9}$$

The upper danger limit is greater than 1.996, the upper specification limit, but is less than 1.997, which exceeds the upper specification limit by one thousandth of an inch. Correspondingly, the lower danger limit is less than 1.990 but greater than 1.989. Thus the control is again shown to be satisfactory.

We next find the control limits for the chart for ranges. On this chart the location of the center line is determined by the value of $\bar{R}$ given in (4). To locate the control limits we calculate the value of Y_R, the distance from the center line to the upper control limit. For this we substitute in (19), Art. 22, the value of $\bar{R}$ given in (4), and

$$D = 1.282 \tag{10}$$

given in (20), Art. 22, for a sample of 4, and obtain

$$Y_R = D\bar{R} = 1.282 \times 0.0023 = 0.0029 \tag{11}$$

to four decimals. Using the condensed form for locating the control limits, with the value of $\bar{R}$ as given in (4), and of Y_R in (11), we may write

$$\begin{aligned} \bar{R} \pm Y_R = \bar{R} \pm D\bar{R} &= 0.0023 \pm 0.0029 \\ &= \begin{cases} 0.0052 & \text{U.C.L.} \\ 0.0000 & \text{L.C.L.} \end{cases} \end{aligned} \tag{12}$$

On the chart for ranges of Fig. 23, the upper control limit is shown as a dashed line from 30 to 60. The lower control limit coincides with the base line.

The charts show that the process is under satisfactory control and that this control has been maintained for the period covered by the past 30 samples. The control values can now be adopted as standard and the limits can be projected forward and used for controlling quality during future production.

If control continues as satisfactory as it was from samples 31 to 60 then it will be unnecessary to inspect so frequently. However, samples of 4 should be inspected often enough to show that control is maintained, and samples should be inspected more frequently at any time that control appears questionable.

The first 30 samples showed that the chart for averages provided a more sensitive index of the state of control than the chart for ranges did. The chart for ranges did not show any point outside of control whose corresponding point on the chart for averages was not also outside of control, while the chart for averages showed several more points that were outside of control limits. Since the condition of control can be followed satisfactorily from the chart for averages alone, the $\bar{X}$-chart would be continued for control during production, or for process control, while the R-chart would be discontinued. Hence, as indicated in the space for notes on Fig. 23, the value of $\bar{\bar{X}}$ in (3) and those for the location of control limits given in (7) were adopted as standard values.

With the illustration of this article, as with the one in Art. 22, a substitute formula, instead of the formula of statistical methods, was used to obtain the control limits. With this substitute method the formula $Y_A = A_2\bar{R}$ given in (6) was used for computing the distance from the center line to either control limit on the chart for averages. In the expression on the right in this formula the factor A_2 is a table value which depends upon the sample size, while the second factor $\bar{R}$ is obtained from the process by averaging the ranges for a large number of samples. Since in any chart application the sample size should be kept constant, the value of A_2 cannot change. However, the value of $\bar{R}$, and hence of Y_A, could be different for different groupings of samples. For example, the value of $\bar{R}$ computed for the first 25 samples of the tabulation on sheet 2 of Table 23 is 0.0022, and for the last 25 the value of $\bar{R}$ is 0.0024, each to four decimals. As the sample groupings are obtained by chance the value of $\bar{R}$ resulting from any chance grouping should give a value of Y_A that would be satisfactory for locating control limits, provided the process is under satisfactory control.

The value of Y_A must meet the test that danger limits obtained from it must be satisfactorily located with reference to the specifications. The danger limits obtained above in (9) corresponded to a value of 0.0023 for $\bar{R}$, and these limits were found to be satisfactory. Would the value of $\bar{R} = 0.0024$ for the last 25 samples be too large, or how large could $\bar{R}$ be and not place the danger limits too far from the center line?

This largest value which $\bar{R}$ can have may be readily determined because the limits for it are fixed by specifications. For the present illustration the limits of Z are such that the lower danger limit must be above 1.989 and the upper danger limit must be below 1.997; 1.990 being the smallest acceptable value, 1.996 being the largest, and measurements being taken to the nearest thousandth of an inch. This means that Z must be less than 0.004. In the formula $Y_A = Z/\sqrt{n}$, appearing in (1), we put $n = 4$ and $Z = 0.004$ and see that Y_A must be less than 0.002. Now to obtain the limit for $\bar{R}$ we note that $\bar{R} = Y_A/A_2$, from the formula appearing in (6), and this formula with $Y_A = 0.002$ and $A_2 = 0.729$ gives $\bar{R} = 0.00274$. Hence, a value of $\bar{R}$ of 0.0027 or less would keep the danger limits within satisfactory locations.

24. Comments on Distribution

It should be remarked that not all applications will develop to a state of control, and such very satisfactory control, as was the case with the illustration with the gear of Art. 23. Certainly not in such a short time. That illustration was taken from a plant where everyone was trained to maintain high standards of quality. Furthermore, the process for manufacturing the gear had been in progress for some time and the quality had been carefully controlled though not by use of statistical methods. The excessive trouble with the first 30 was unusual. Then too, as stated in Art. 23, a few slight adjustments and omissions were made in order to complete the preliminary stages within the limits of Fig. 23.

In many applications there will be several causes for trouble and it may take a rather long period for the engineering department to track down all causes for trouble and to take necessary measures for correcting them.

Frequently with a control chart analysis, whether with the $\bar{X}$- and R-charts of the present chapter or with the charts of Chap. IV that arise from inspecting by attributes, it may be possible to discard some of the data for the preliminary period when calculating control limits. If the plotted points for preliminary data reveal a lack of control, the data for points outside of control should be carefully examined. If a point falls outside a control limit and there is some engineering explanation of the condition that caused it to be outside of this limit, and if a correction can be made so that that condition will not affect future production, then the measurements for the sample, or samples, for which this is true, can be disregarded in calculating the average and the control limits. Data for all such samples for the preliminary period can be disregarded, and the remaining values can be used for calculating preliminary control limits.

However, care must be exercised not to discard any recorded values unless an engineering reason is found why the conditions which prevailed when the sample was taken were not representative of the normal process, and further, the value should not be discarded unless corrections can be made so that these conditions will not influence future production. A sample should not be discarded just because its plotted point falls outside of a control limit.

After discarding all data that can be disregarded for known engineering reasons the remaining preliminary data should be used and control limits should be computed and located on the charts. If lack of control is then in evidence, a search should be made for causes of trouble, and the search should be continued until one after the other the causes of trouble have been located and corrective action has been taken. As soon as enough improvement has been made, new control limits should be computed and should be drawn on the charts.

When the process has been brought under satisfactory control and continues under this control for a period covered by 25 or 30 samples, standard values for average and control limits, based on these samples, should be computed and adopted. These control limits should be located on the charts and should be projected forward for use during future production, as was done with Fig. 23.

A study of Fig. 24*a* and of Figs. 19*a*, 19*b*, 19*c*, and 19*d* will help the reader to gain a better understanding of standard values and of acceptable limits for satisfactory control. The indicated percentages shown on Fig. 24*a* will supplement statements regarding distributions that were made in Arts. 15, 17, and 22*g*.

A normal, symmetrical frequency distribution has been shown in Fig. 24*a*, with the center line ℄, and the upper and the lower danger limits, U.D.L. and L.D.L., located and identified. On this chart the percentages of the variables that should fall between a few specified limits have been indicated. The frequency curve has been shown with end points A and B, where A represents the smallest value that is acceptable under specifications, and B the largest. The curve, the center line, the control limits, and the percentage distributions are as shown provided that the distribution is a normal, symmetrical one. If the control is satisfactory then the danger limits, and the limits within which variables should fall to satisfy specification requirements, should be related as shown on Fig. 19*a*. By comparing Figs. 19*b*, 19*c*, and 19*d* with Fig. 24*a* the reader should be able to determine approximately what percentage of scrap would be produced in each case.

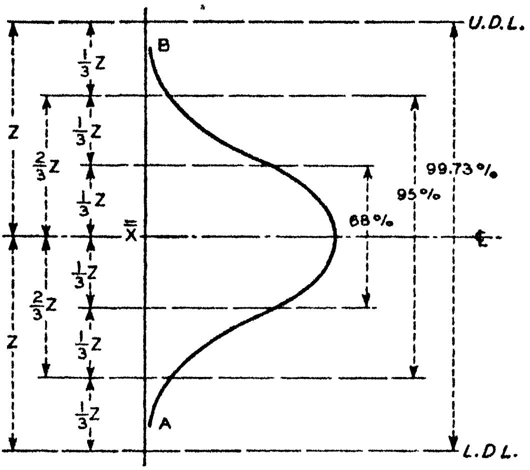


FIG. 24*a*.

Since on Fig. 24*a* U.D.L. and L.D.L. are upper and lower limits for variables that represent individual items of a controlled process, and 99.73 per cent of the variables should be within these limits, we see that not more than 3 in 1,000 individual items should have measurements that are outside of these limits. Or we may say that by chance we could expect about 0.13 per cent to fall outside of either danger limit. The general concentration of the variables about the central, or normal, value is seen by the other percentage distributions which are shown. From the figure we see that approximately 95 per cent fall within the range $\bar{\bar{X}} + \frac{2}{3}Z$ down to $\bar{\bar{X}} - \frac{2}{3}Z$, and 68 per cent within $\bar{\bar{X}} + \frac{1}{3}Z$ and $\bar{\bar{X}} - \frac{1}{3}Z$.

Obviously any condition that is represented by a frequency curve will also be represented on a control chart for the same data, if the control chart is based on a sufficiently large number of samples. From the position of the danger limits with reference to specification limits, the percentage of the items that represent scrap can be found approximately.

It may sometimes be economically unsound to make changes that would be necessary for improving the process and for reducing scrap. If the percentage of scrap is low, it might be more economical to cull out the scrap at the assembly line, when safe to do so, than to undertake expensive alterations.

When a frequency distribution is not symmetrical, we may have a curve such as shown in Fig. 24*b*. This is called a "skewed frequency curve," and obviously we may expect many different kinds of skewed curves. It is much more difficult to express the percentage distributions for skewed curves and we shall make no attempt to show any here. If the distribution is not too far from a symmetrical one, we can use a control chart with limits computed as for a symmetrical distribution. We would have to check both danger limits and if each fell within its own specification limits, the control chart would be acceptable.

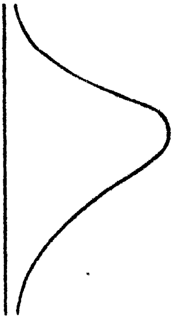


FIG. 24*b*.

25. Demonstration with Dice

The three most important influences upon the quality of manufactured items are man, machine, and material. Unsatisfactory quality may result from poor workmanship, or from a sick machine, or because of inferior material, or from any two or all three of these influences. The way in which trouble may arise from any one or more of these and the manner in which such trouble is reflected upon a control chart, or upon a frequency distribution, is often demonstrated by use of three dice. Since this is an interesting demonstration, we shall give a brief description of one form of it so that the reader may make a demonstration for himself, and we shall show how the control chart is affected when something goes wrong with one or more of the factors.

For the demonstration we should have two sets of dice. One set, which we shall refer to as the regular set, will have three ordinary dice of the same size and appearance, but will be of three different colors. The other set, which we shall refer to as the phony set, will consist of three dice of the same size, color, and general appearance as the first set, but each die will have two opposite faces marked[1] with 6, two with 5, and two with 4.

With each set of dice select one color to represent man, another to represent machine, and the third to represent material.

If the three regular dice are rolled a large number of times the frequency distribution of the total for each roll should be the same as if rolling any three ordinary unloaded dice. The numbers appearing with greatest frequency should be 10 or 11, those with next frequency should be 9 and 12, and on down and up to 3 and 18, which should have the least frequency, and the average should be between 10 and 11. The theoretical frequency distribution for 216 tosses is given in Table 25. After the three

TABLE 25.—THEORETICAL DISTRIBUTION IN ROLLING THREE DICE

Total, each roll X	Theoretical frequency in 216 rolls f
3	1
4	3
5	6
6	10
7	15
8	21
9	25
10	27
11	27
12	25
13	21
14	15
15	10
16	6
17	3
18	1
Total........	216

---

[1] Of course there are many other ways in which the phony dice could be marked and still make them serviceable for the demonstration. There could be two faces with 1, two with 2, and two with 3, which would have the same effect on the low side as the ones given have on the high side. Or all faces could be marked with a 6 and then the change in the frequency distribution would become apparent more quickly, but it would be easier for the presence of the phony dice to be detected by eye.

regular dice have been rolled often enough to establish a satisfactory frequency table, or to show that the average is around 10 or 11, then substitute a phony die (or two of them, or all three), and note how soon the changed frequency distribution will reveal its presence.

When instructing a group, this makes a very convincing demonstration. The instructor should slip in a phony die in place of a regular one without the group's being aware of the substitution. The changed average, or the changed distribution, should become evident to the members of the group very shortly after the substitution has been made. It then becomes easy to show that, similarly, poor workmanship, a faulty machine, or inferior material will be revealed by the behavior of a frequency distribution or a control chart.

26. Use of Disks to Represent a Manufacturing Process

Since the variables in a controlled manufacturing process obey the laws of chance, a controlled process can be represented quite satisfactorily by use of appropriately numbered disks, or chips, with numbers to represent measurements of the characteristic to be considered. Each disk will bear one measurement, and the measurements on the various disks will range from the lowest to the highest value allowed by specifications. There should be enough disks marked so that the frequencies will conform to a normal distribution; we shall use 200 with each of the distributions that we introduce.

Let us take an example to illustrate how to use the disks for building a control chart. For the example let us assume that we are to make an $\bar{X}$-chart and an R-chart for a portion of a shaft for which specifications call for a diameter of 1.644 inches ± 0.003 inch. In Table 26*a* we have given a theoretical, symmetrical distribution of dimensions within this range for 200 shafts.

We should now mark 3 disks[1] with 1.647, 19 with 1.646, 47 with 1.645, and so on for the other entries in the table. When we read a number from a disk we should think of it as representing the diameter of a shaft which we had just measured to the nearest thousandth of an inch.

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[1] Ordinary disks that are used for marking prices, stock numbers, etc., are very useful for the demonstration. Or chips, which can be bought from manufacturers of chips, can be used.

The marking as just indicated will give the most satisfactory picture of the actual measurements of shafts. However, a substitute marking, which may be used if preferred, is shown in the third column of Table 26*a*. This system of marking is much simpler than the other and will be satisfactory if it is remembered that 0 represents a diameter of 1.644 inches, that each plus number, expressed in thousandths, is to be added to 1.644, and that the numerical value of each minus number, expressed in thousandths, is to be subtracted from 1.644. In other words the substitute figures in the third column are to be thought of as representing the corresponding diameters given in the first column.

Table 26*a*.—Theoretical Frequency Distribution for the Diameters of 200 Shafts

Diameter, inches X	Frequency in 200 f	Substitute for actual diameters X'
1.647	3	+3
1.646	19	+2
1.645	47	+1
1.644	62	0
1.643	47	−1
1.642	19	−2
1.641	3	−3
Total	200	

This substitute marking is particularly useful because the disks so marked may be used to represent any other distribution for which the spread ranges from +3 to −3. For example, we could use this marking and could select samples for the gear of Table 23 if we took 0 to represent 1.993, and if we added the plus numbers in thousandths to 1.993 and subtracted the negative ones from 1.993. The correspondence between the two sets of numbers for this case would be as shown in Table 26*b*.

We shall now give a brief description of the demonstration using disks marked with the substitute numbers of Table 26*a*, and then the reader can make as many charts as he may wish. We shall follow this description by reproducing a table and a chart that resulted from a class demonstration. Then in the exercises at the end of the chapter the reader will be given further opportunities to make applications using the frequency distribution of Table 26*a* and other frequency distributions that will be given in Table 28*a*.

For the demonstration we place the disks in a box, or other suitable container, mix them thoroughly, and draw a random sample from the box. Either type of marking of Table 26*a* may be used on these disks; we shall use the substitute marking for this illustration. In order to use the formulas of Art. 22, the sample size should be 10 or less, but preferably 4, 5, or 6. We shall describe the procedure for a sample size of 5 but the reader should experiment with others. Having taken a sample, try to imagine that you are in a factory and that a sample has been taken from the production line and that the actual measurements of the diameters correspond to those obtained from the numbers on the disks of your sample. Enter the figures from the disks on a form like that used for Table 23, or any other satisfactory form, compute the average and range for the sample, record these in the appropriate spaces, and mark the corresponding points on the charts.

TABLE 26*b*.—THEORETICAL FREQUENCY DISTRIBUTION FOR O.D. OF 200 GEARS

O.D. of gears in inches X'	Frequency in 200 f	Substitute for actual O.D. X
1.996	3	+3
1.995	19	+2
1.994	47	+1
1.993	62	0
1.992	47	−1
1.991	19	−2
1.990	3	−3
Total	200	

Now replace the disks in the box, thoroughly mix again, take a second sample, and go through the same procedure as for the first sample. Repeat the procedure until 25 or 30 samples have been recorded, averages and ranges have been computed, and points have been plotted on the charts. From the data now recorded you can compute the averages $\bar{\bar{X}}$ and $\bar{R}$, and the control limits for the $\bar{X}$-chart and the R-chart, and hence complete the charts.

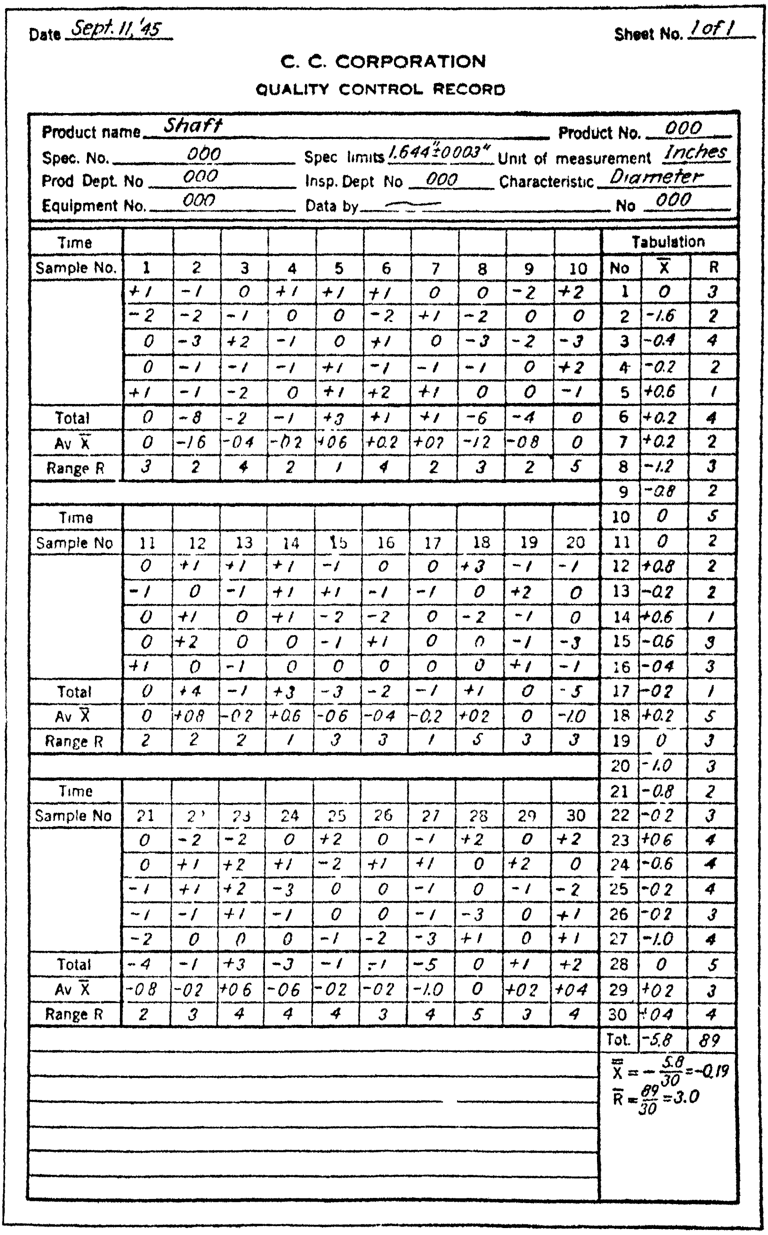
The recorded data, the averages, and the control charts will give a very satisfactory representation of the same things from an actual application in a factory. There are some definite differences between the data obtained and the charts drawn through the use of disks and those obtained from a practical application, but the charts should be essentially alike. The chief difference is due to using the same good symmetrical theoretical distribution from which to take each sample when using disks, while for the shop application we take our sample from a different lot each time, and the distribution of each lot is only approximately symmetrical. A second big difference is represented by the total time elapsed and the number of persons involved. With the disks one person can take all 30 samples in less than that many minutes. In a practical application, if 2 samples were taken every hour then operators and inspectors from at least two shifts would make and inspect the shafts that were selected.

The reader can think of other differences, but he should realize that the selection of data by use of disks should very closely represent the actual collection of data from a process in the factory. If he makes charts by using disks for obtaining his data, the practice he will get in recording data, in computing averages and plotting points on the charts, in computing control limits and locating them on the charts, and in reading and interpreting the charts, should enable him to make and use charts for a practical application.

In Table 26*c* we have recorded the data and the computations representing 30 drawings of samples of 5 disks from a lot of 200 with the distribution of Table 26*a*. The disks were placed in a flour sifter and thoroughly mixed before each drawing, the disks from each sample being replaced before proceeding to pick the next one.

The total for the first sample is 0, and hence the average is 0, while the range is 3. These values are entered opposite 1 in the tabulation columns. We should think of this sample, and of all others, in terms of the product which we are representing by the

TABLE 26c.



Date *Sept. 11, '45* Sheet No. *1 of 1*

C. C. CORPORATION

QUALITY CONTROL RECORD

Product name *Shaft* Product No. *000*

Spec. No. *000* Spec limits *1.644"±0.003"* Unit of measurement *Inches*

Prod Dept. No *000* Insp. Dept No *000* Characteristic *Diameter*

Equipment No. *000* Data by ______ No *000*

Time											Tabulation		
Sample No.	1	2	3	4	5	6	7	8	9	10	No	$\bar{X}$	R
	+1	-1	0	+1	+1	+1	0	0	-2	+2	1	0	3
	-2	-2	-1	0	0	-2	+1	-2	0	0	2	-1.6	2
	0	-3	+2	-1	0	+1	0	-3	-2	-3	3	-0.4	4
	0	-1	-1	-1	+1	-1	-1	-1	0	+2	4	-0.2	2
	+1	-1	-2	0	+1	+2	+1	0	0	-1	5	+0.6	1
Total	0	-8	-2	-1	+3	+1	+1	-6	-4	0	6	+0.2	4
Av $\bar{X}$	0	-16	-04	-02	+06	+0.2	+02	-12	-08	0	7	+0.2	2
Range R	3	2	4	2	1	4	2	3	2	5	8	-1.2	3
											9	-0.8	2
Time											10	0	5
Sample No	11	12	13	14	15	16	17	18	19	20	11	0	2
	0	+1	+1	+1	-1	0	0	+3	-1	-1	12	+0.8	2
	-1	0	-1	+1	+1	-1	-1	0	+2	0	13	-0.2	2
	0	+1	0	+1	-2	-2	0	-2	-1	0	14	+0.6	1
	0	+2	0	0	-1	+1	0	0	-1	-3	15	-0.6	3
	+1	0	-1	0	0	0	0	0	+1	-1	16	-04	3
Total	0	+4	-1	+3	-3	-2	-1	+1	0	-5	17	-02	1
Av $\bar{X}$	0	+08	-02	+0.6	-06	-04	-0.2	+02	0	-1.0	18	+0.2	5
Range R	2	2	2	1	3	3	1	5	3	3	19	0	3
											20	-1.0	3
Time											21	-0.8	2
Sample No	21	22	23	24	25	26	27	28	29	30	22	-02	3
	0	-2	-2	0	+2	0	-1	+2	0	+2	23	+06	4
	0	+1	+2	+1	-2	+1	+1	0	+2	0	24	-0.6	4
	-1	+1	+2	-3	0	0	-1	0	-1	-2	25	-02	4
	-1	-1	+1	-1	0	0	-1	-3	0	+1	26	-02	3
	-2	0	0	0	-1	-2	-3	+1	0	+1	27	-1.0	4
Total	-4	-1	+3	-3	-1	-1	-5	0	+1	+2	28	0	5
Av $\bar{X}$	-08	-02	+06	-06	-02	-02	-1.0	0	+02	+04	29	+02	3
Range R	2	3	4	4	4	3	4	5	3	4	30	+04	4
											Tot.	-5.8	89

$\bar{\bar{X}} = -\frac{5.8}{30} = -0.19$

$\bar{R} = \frac{89}{30} = 3.0$

disks. Thus we think of measuring the diameter of each shaft and of averaging the measurements to obtain 0. This of course means that in the first sample the average diameter was 1.644

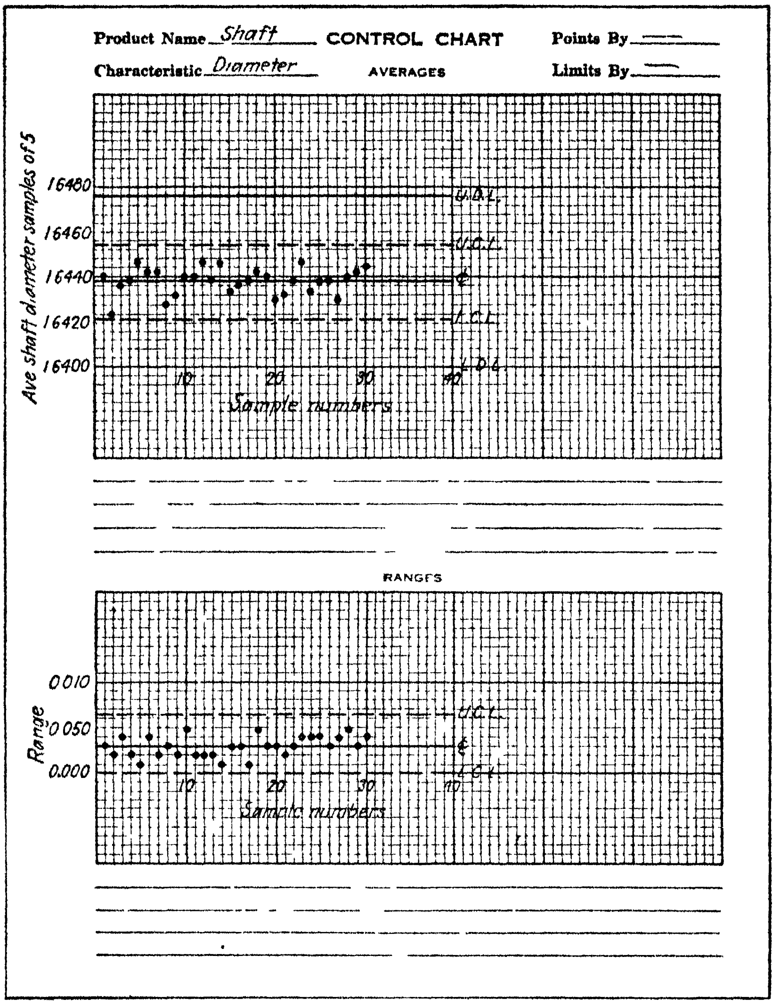


Fig 26

inches. The range of 3 corresponds to an actual range of 0.003 inch. For the second sample the average of −1.6 actually represents an average shaft diameter of 1.644 inches − 0.0016 inch = 1.6424 inches.

The charts could be drawn with the vertical scale in terms of the substitute numbers in place of actual diameters. However, there is not the same cause for using the substitute numbers for the charts that there is with the recorded data. With recorded data the substitute numbers give great savings in time and labor in making computations. With the charts the scales are permanently identified on the figure, and if the substitute numbers represent the characteristic measured as convincingly as they should, there will be no difficulty in plotting points in terms of actual measurements, even though in the tabulation $\bar{X}$ and R values are expressed in terms of substitute numbers.

Since the charts are so much more useful and appropriate when the vertical scales show actual measurements than when substitute numbers are shown, we have used the scales in this way with Fig. 26.

We must now obtain the locations of the center line and the control limits. For this we shall use the formulas of Art. 22.

In terms of the substitute numbers the center line for the chart for averages is located by the value of $\bar{\bar{X}}$ given in Table 26*c*. To one decimal this is

$$\bar{\bar{X}} = -0.2 \tag{1}$$

To locate the control limits on this chart we substitute in (4), Art. 22, the value of

$$\bar{R} = 3 \tag{2}$$

from Table 26*c*, and $A_2 = 0.577$, for a sample size of 5 as given in (6), Art. 22 (or as given in Table III), and find

$$Y_A = A_2\bar{R} = 0.577 \times 3 = 1.731 = 1.7 \tag{3}$$

to one decimal. The control limits may now be written in the combined and condensed form of (10), Art. 22. Hence we have

$$\bar{\bar{X}} \pm Y_A = \bar{\bar{X}} \pm A_2\bar{R} = -0.2 \pm 1.7 = \begin{cases} 1.5 & \text{U.C.L.} \\ -1.9 & \text{L.C.L.} \end{cases} \tag{4}$$

for the locations of the control limits in terms of the substitute numbers, or in thousandths of an inch.

To obtain the locations of the center line and the control limits in terms of actual measurements, or in other words in inches, we first note that we may interpret the result in (1) as giving the average value for the diameters of the shafts as 0.0002 inch less than 1.6440 inches. Thus, for the location of the center line in inches, we have

$$\bar{\bar{X}} = 1.6440 - 0.0002 = 1.6438 \tag{5}$$

To locate the control limits we multiply the value of Y_A in (3) by 0.001 and obtain 0.0017 as the spread from the center line to the control limits. This gives

$$\begin{aligned} \bar{\bar{X}} \pm Y_A &= \bar{\bar{X}} \pm A_2\bar{R} = 1.6438 \pm 0.0017 \\ &= \begin{cases} 1.6455 & \text{U.C.L.} \\ 1.6421 & \text{L.C.L.} \end{cases} \end{aligned} \tag{6}$$

The value $\bar{R} = 3$, given in (2), is used to locate the center line for the chart for ranges. Substituting this value of $\bar{R}$ and the value of $D = 1.114$ for a sample of 5 [given in (20), Art. 22 or in Table III], in (19), Art. 22, we find

$$Y_R = D\bar{R} = 1.114 \times 3 = 3.342 = 3.3 \tag{7}$$

to one decimal. Then the location of the upper control limit on the chart for ranges, obtained by adding this value of Y_R to the value of $\bar{R}$, is

$$\bar{R} + Y_R = 3 + 3.3 = 6.3 \tag{8}$$

Since the sample size is 5 we use the second formula of (18), Art. 22, and take zero for the lower control limit.

In terms of actual measurements, or inches, the location of the center line and the control limits on the chart for ranges are

(9) 0.003 for the center line

(10) 0.0063 for upper control limit

(11) 0.0000 for lower control limit

When the lines giving the location of control limits are drawn on the charts it is found that the process represented by this demonstration is under control.

To see if the control is satisfactory we next locate the danger limits on the chart for averages. Using formula (13), Art. 22, with $Y_A = 1.7$ and $n = 5$ we find

$$Z = \sqrt{n}Y_A = \sqrt{5} \times 1.7 = 2.236 \times 1.7 = 3.8 \tag{12}$$

to one decimal. In terms of substitute numbers the danger limits are then given by

$$\bar{\bar{X}} \pm Z = -0.2 \pm 3.8 = \begin{cases} 3.6 & \text{U.D.L.} \\ -4.0 & \text{L.D.L.} \end{cases} \tag{13}$$

In terms of actual measurements we may write these limits as

$$\bar{\bar{X}} \pm Z = 1.6438 \pm 0.0038 = \begin{cases} 1.6476 & \text{U.D.L.} \\ 1.6400 & \text{L.D.L.} \end{cases} \tag{14}$$

Since the upper danger limit lies between 1.6470 and 1.6480, and the lower limit is at 1.6400, the control is satisfactory. The level is slightly low, and if it were any lower the control would be unsatisfactory.

27. Short Cuts

The use of substitute numbers in the last article introduced one method of reducing the labor of computations. Those readers who are experienced in making computations have probably noticed many other opportunities for saving time and labor. These laborsaving devices are generally classified as short cuts, and anyone who can make use of any short cut, and can still obtain accurate results, is encouraged to do so. We have avoided the use of short cuts, excepting with substitute numbers, because their use is apt to introduce an element of confusion for those who are not thoroughly practiced in computations and in the use of formulas.

28. Some Distributions for Practice

In Table 28*a* we have given some distributions that can be used for a great many applications in building and interpreting control charts. The numbers are given in the substitute form of the tables in Art. 26. They can be used to represent any characteristic of any product if the units of measurement are expressible in a corresponding number of divisions.

A very wide spread is shown for the third distribution. Such a spread is entirely possible, but it is given here primarily to enable certain effects to be produced in the exercises.

TABLE 28a.—SUGGESTED FREQUENCY DISTRIBUTIONS FOR SETS OF DISKS FOR SAMPLING FOR VARIABLES

Numbers X	Frequency in 200			
	Set A f	Set B f	Set C f	Set D f
10			1	
9			1	
8			1	
7		1	3	
6		3	5	
5	1	10	8	
4	3	23	12	
3	10	39	16	3
2	23	48	19	19
1	39	39	22	47
0	48	23	24	62
−1	39	10	22	47
−2	23	3	19	19
−3	10	1	16	3
−4	3		12	
−5	1		8	
−6			5	
−7			3	
−8			1	
−9			1	
−10			1	
Total....	200	200	200	200

Four sets of disks should be marked with the numbers in each set having the frequencies indicated for that set. Set A and Set B have the same spreads but different averages, while Set A and Set C have the same averages but widely different spreads. The spread in Set D is not so wide as the others. This distribution (in Set D) has been used with several illustrations and may be more satisfactory for representing practical applications than the others are.

EXERCISES

1. Make an $\bar{X}$-chart and an R-chart using 200 disks marked as in Set A of Table 28a. Take 30 samples of 4 and make all necessary computations for averages and for locations of control limits, and plot center lines and control limits on the charts. Project the lines forward for use with Exercises 2 and 3.

2. Take samples of 4 using 200 disks marked as in Set B and plot points for the averages and the ranges of these samples on the charts of Exercise 1. Continue to draw samples from this set of disks until several points have fallen outside the upper control limit on the chart for averages, and until it is demonstrated that the points all have a tendency to be high on the chart for averages. With this set of disks, points on the chart for ranges should fall satisfactorily inside of control limits.

3. Take samples of 4 from 200 disks marked as in Set C and plot points for the averages and the ranges of these samples on the charts of Exercise 1. Continue to draw samples from this set of disks until several points have fallen outside of either the upper or the lower control limit on the chart for averages, and of the upper limit on the chart for ranges. The change in the condition of control should show up more definitely on the R-chart than on the $\bar{X}$-chart, but it should also be evidenced by some points falling outside of control limits on the $\bar{X}$-chart.

4. Make an $\bar{X}$-chart and an R-chart using 200 disks marked as in Set B of Table 28a. Take 30 samples of 5 and make all necessary computations for averages and for locating control limits on the charts. Project the lines forward for use with Exercises 5 and 6.

5. Take samples of 5 using 200 disks marked as in Set A and plot points for the averages and the ranges of these samples on the charts of Exercise 4. Continue to draw samples from this set of disks until several points have fallen below the lower control limit on the chart for averages, and until it is demonstrated that the points on this chart all have a tendency to be low. With this set of disks, points on the chart for ranges should fall satisfactorily inside of control limits.

6. Take samples of 5 from 200 disks marked as in Set C and plot points for the averages and the ranges of these samples on the charts of Exercise 4. Continue to draw samples from this set of disks until several points have fallen outside of a control limit on the chart for averages, and of the upper control limit on the chart for ranges. Again the change in the condition of control should show up more definitely on the R-chart than on the $\bar{X}$-chart, but there should be a general trend toward the lower control limit of the points on the chart for averages. On the chart for averages an occasional point could fall outside the upper control limit, but there should be more points below the lower control limit than above the upper.

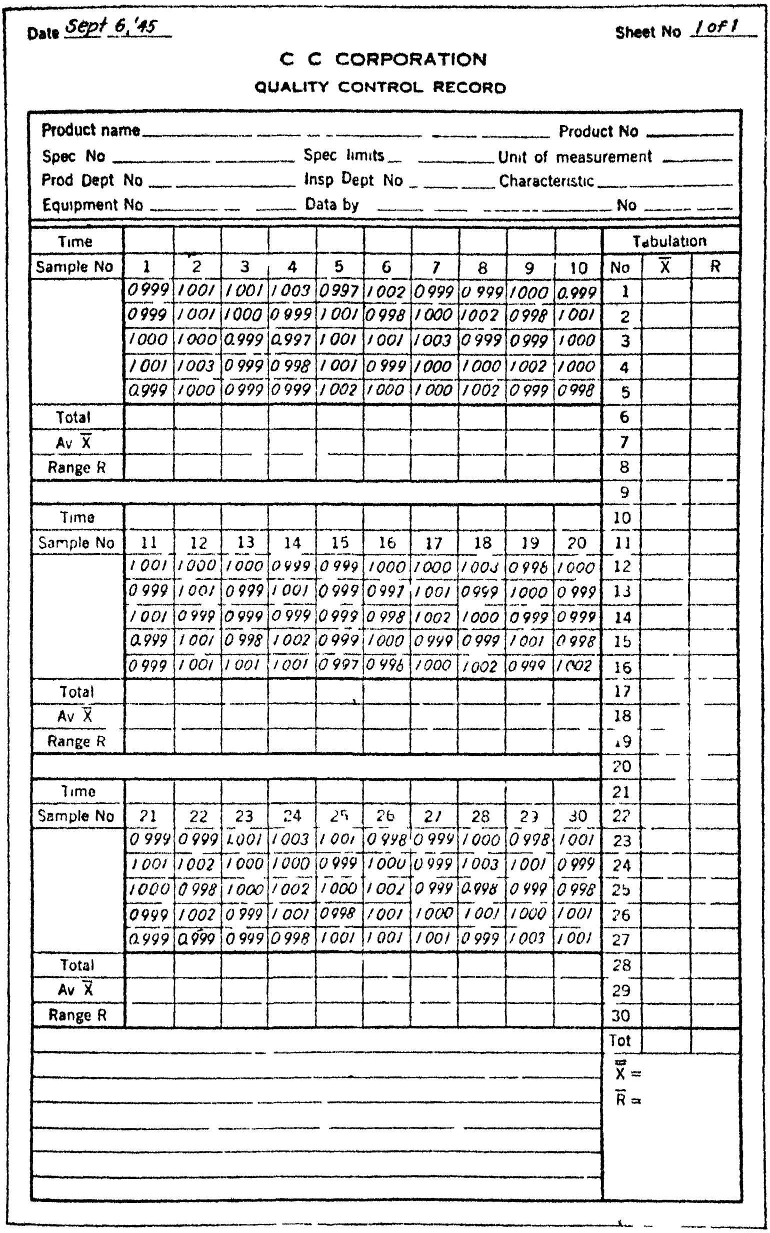
7. Compute the averages and ranges for the data of the 30 samples of 5 recorded in Table 28*b* (given below) and enter them in the tabulation columns. Plot the points on an $\bar{X}$-chart and an R-chart, and make the necessary computations for locating the control limits. Draw the center lines and the lines for control limits. Discuss the question of control of quality to meet specifications of 1.000 ± 0.003.

The data for this table were obtained by taking samples of 5 disks from a lot of 200. The distribution of Set *D*, Table 28*a*, was used, with the numbers marked on the disks ranging from 0.997 to 1.003, with 0 representing 1.000.

The reader should think of the data as representing a characteristic of some product in his plant and should identify his solution accordingly. He could, of course, change the unit of measurement if necessary to suit his application, provided he kept an allowed range of ± 3 from his central value.

8. Follow the instructions given for Exercise 7 but using the data of Table 28*c*. The data for this table were obtained by taking samples of 4 from the same lot as used for Exercise 7.

TABLE 28b.



Date Sept 6,'45

Sheet No 1 of 1

C C CORPORATION

QUALITY CONTROL RECORD

Product name ____ Product No ____

Spec No ____ Spec limits ____ Unit of measurement ____

Prod Dept No ____ Insp Dept No ____ Characteristic ____

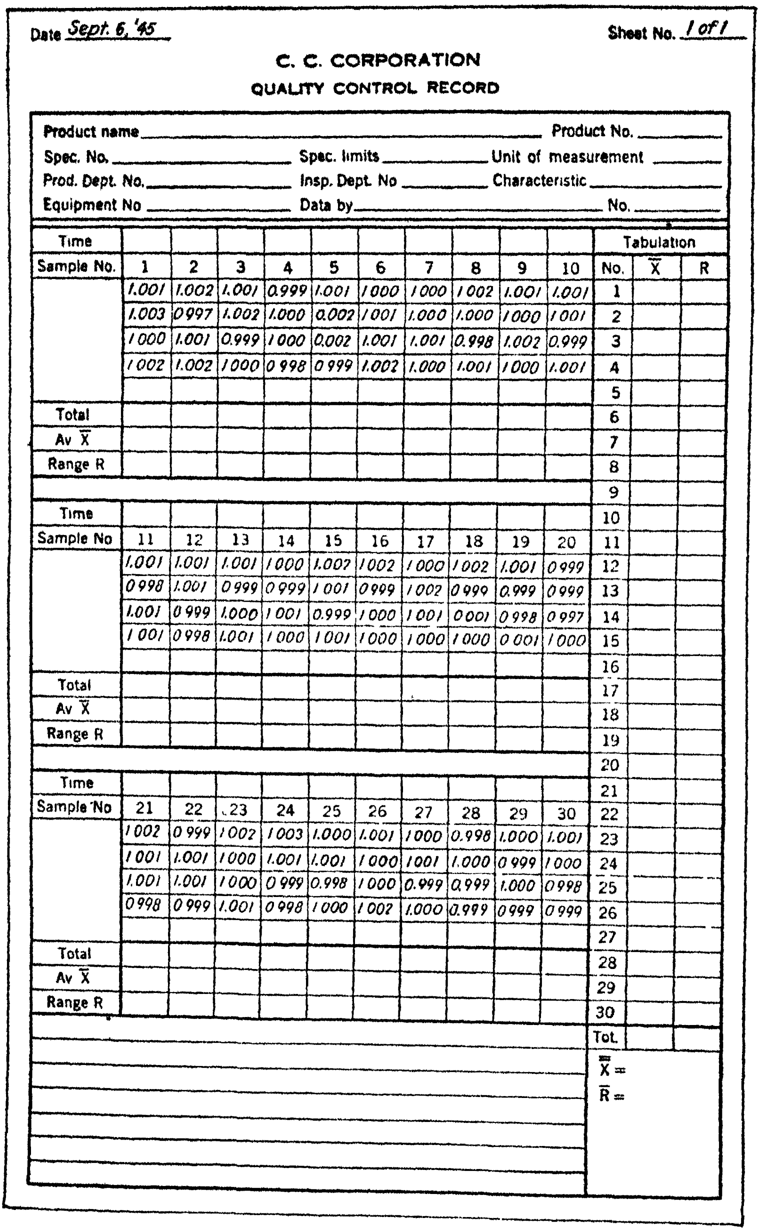
Equipment No ____ Data by ____ No ____

Time											Tabulation		
Sample No	1	2	3	4	5	6	7	8	9	10	No	$\bar{X}$	R
	0.999	1.001	1.001	1.003	0.997	1.002	0.999	0.999	1.000	0.999	1		
	0.999	1.001	1.000	0.999	1.001	0.998	1.000	1.002	0.998	1.001	2		
	1.000	1.000	0.999	0.997	1.001	1.001	1.003	0.999	0.999	1.000	3		
	1.001	1.003	0.999	0.998	1.001	0.999	1.000	1.000	1.002	1.000	4		
	0.999	1.000	0.999	0.999	1.002	1.000	1.000	1.002	0.999	0.998	5		
Total											6		
Av $\bar{X}$											7		
Range R											8		
											9		
Time											10		
Sample No	11	12	13	14	15	16	17	18	19	20	11		
	1.001	1.000	1.000	0.999	0.999	1.000	1.000	1.003	0.996	1.000	12		
	0.999	1.001	0.999	1.001	0.999	0.997	1.001	0.999	1.000	0.999	13		
	1.001	0.999	0.999	0.999	0.999	0.998	1.002	1.000	0.999	0.999	14		
	0.999	1.001	0.998	1.002	0.999	1.000	0.999	0.999	1.001	0.998	15		
	0.999	1.001	1.001	1.001	0.997	0.996	1.000	1.002	0.999	1.002	16		
Total											17		
Av $\bar{X}$											18		
Range R											19		
											20		
Time											21		
Sample No	21	22	23	24	25	26	27	28	29	30	22		
	0.999	0.999	1.001	1.003	1.001	0.998	0.999	1.000	0.998	1.001	23		
	1.001	1.002	1.000	1.000	0.999	1.000	0.999	1.003	1.001	0.999	24		
	1.000	0.998	1.000	1.002	1.000	1.002	0.999	0.998	0.999	0.998	25		
	0.999	1.002	0.999	1.001	0.998	1.001	1.000	1.001	1.000	1.001	26		
	0.999	0.999	0.999	0.998	1.001	1.001	1.001	0.999	1.003	1.001	27		
Total											28		
Av $\bar{X}$											29		
Range R											30		
											Tot		

$\bar{\bar{X}}$ =

$\bar{R}$ =

TABLE 28c.



Date *Sept. 6, '45* Sheet No. *1 of 1*

C. C. CORPORATION

QUALITY CONTROL RECORD

Product name ____ Product No. ____

Spec. No. ____ Spec. limits ____ Unit of measurement ____

Prod. Dept. No. ____ Insp. Dept. No ____ Characteristic ____

Equipment No ____ Data by ____ No. ____

Time										
Sample No.	1	2	3	4	5	6	7	8	9	10
	1.001	*1.002*	*1.001*	*0.999*	*1.001*	*1.000*	*1.000*	*1.002*	*1.001*	*1.001*
	1.003	*0.997*	*1.002*	*1.000*	*0.002*	*1.001*	*1.000*	*1.000*	*1.000*	*1.001*
	1.000	*1.001*	*0.999*	*1.000*	*0.002*	*1.001*	*1.001*	*0.998*	*1.002*	*0.999*
	1.002	*1.002*	*1.000*	*0.998*	*0.999*	*1.002*	*1.000*	*1.001*	*1.000*	*1.001*
Total										
Av $\overline{X}$										
Range R										

Time										
Sample No	11	12	13	14	15	16	17	18	19	20
	1.001	*1.001*	*1.001*	*1.000*	*1.002*	*1.002*	*1.000*	*1.002*	*1.001*	*0.999*
	0.998	*1.001*	*0.999*	*0.999*	*1.001*	*0.999*	*1.002*	*0.999*	*0.999*	*0.999*
	1.001	*0.999*	*1.000*	*1.001*	*0.999*	*1.000*	*1.001*	*0.001*	*0.998*	*0.997*
	1.001	*0.998*	*1.001*	*1.000*	*1.001*	*1.000*	*1.000*	*1.000*	*0.001*	*1.000*
Total										
Av $\overline{X}$										
Range R										

Time										
Sample No	21	22	23	24	25	26	27	28	29	30
	1.002	*0.999*	*1.002*	*1.003*	*1.000*	*1.001*	*1.000*	*0.998*	*1.000*	*1.001*
	1.001	*1.001*	*1.000*	*1.001*	*1.001*	*1.000*	*1.001*	*1.000*	*0.999*	*1.000*
	1.001	*1.001*	*1.000*	*0.999*	*0.998*	*1.000*	*0.999*	*0.999*	*1.000*	*0.998*
	0.998	*0.999*	*1.001*	*0.998*	*1.000*	*1.002*	*1.000*	*0.999*	*0.999*	*0.999*
Total										
Av $\overline{X}$										
Range R										

Tabulation

No.	$\overline{X}$	R
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		
26		
27		
28		
29		
30		
Tot.		

$\overline{\overline{X}} =$

$\overline{R} =$

CHAPTER IV

CONTROL CHARTS FOR FRACTION DEFECTIVE, FOR NUMBER OF DEFECTIVES, AND FOR NUMBER OF DEFECTS

29. Inspecting for Attributes

In Art. 20 it was stated that when inspecting for attributes we inspect 100 per cent instead of inspecting small samples as when inspecting for variables. Another important difference between the two methods of inspection results from the fact that when we are inspecting for attributes it is unnecessary to take the actual measurements of each item, while actual measurements are required when inspecting for variables. If we are inspecting for attributes we only need to determine, by visual inspection, by use of a go, no-go-gage, or by any other means, whether or not an article meets specification requirements. If these requirements are satisfied then the article is accepted, otherwise it is rejected.

There are three important charts that result from inspecting for attributes. These are

1. *Chart for fraction defective, or p-chart,*
2. *Chart for number of defectives, or m-chart,*[1]
3. *Chart for number of defects, or c-chart.*

These charts have so many similar features in construction and use that we can give the formulas and the general derivation of the chart for fraction defective and then obtain each of the others by an appropriate modification of this derivation.

---

[1] This chart is usually designated as a *pn*-chart, as will be seen in Chap. V.

30. Definitions

We shall now define some of the terms that will be used with the new charts.

a. Defect.—*A fault which causes an article or an item to fail to meet specification requirements is called a defect.*

b. Defective.—*An item with one or more defects is called a defective item.*

c. Number of Defectives.—The term *number of defectives* needs no definition. We shall use the letter m to denote the number of defectives in a lot and n to represent the total number of items in the lot.

If all lots inspected have the same lot size then the number of defectives m per lot gives an important measure of quality, and a chart based on number of defectives becomes very useful.

d. Number of Defects.—The term *number of defects* is used to represent the number of defects in a sample. The number of defects per sample becomes useful in measuring quality if the possible number of defects per sample may be large compared with the average number. In this connection a sample is usually a single article, such as a designated length of wire, a surface area for metal, etc. We shall follow common practice and designate the number of defects in a sample by c.

e. Fraction Defective.—*The fraction which expresses the ratio of the number of defective items in a lot to the total number of items in the lot is called the fraction defective for the lot.* The fraction defectives for several lots give an important measure of the quality of the production stream from which they are taken. This definition of fraction defective may be expressed through the formula

Fraction defective for a lot

$$\text{(1)} \qquad = \frac{\text{number of defective items in the lot}}{\text{total number of items in the lot}}$$

Following customary practice and designating the fraction defective of a lot by p, and using m and n to denote the number of defective items and the total number of items, respectively, as in c above, we may write (1) as

$$\text{(2)} \qquad p = \frac{m}{n}$$

As an example, let us suppose that we have inspected 500 items and that we found 23 defective ones. For this lot the fraction defective would be

$$p = {}^{23}\!/_{500} = 0.046 \qquad (3)$$

The fraction defective is frequently used in the form of a percentage, which is obtained by multiplying the fraction defective by 100. Thus the fraction defective given in (3) if expressed in the form of a percentage would be

$$p = 4.6 \text{ per cent} \qquad (4)$$

It is often more satisfactory to speak of the "per cent defective" of a lot rather than the "fraction defective," but either expression may be used. However, in the formulas for locating control limits on a chart for fraction defective, the decimal fraction form must be used. Thus with the example above we may say either that the lot is 4.6 per cent defective (or bad) or that its fraction defective is 0.046, but the fraction defective 0.046 would be used in all formulas.

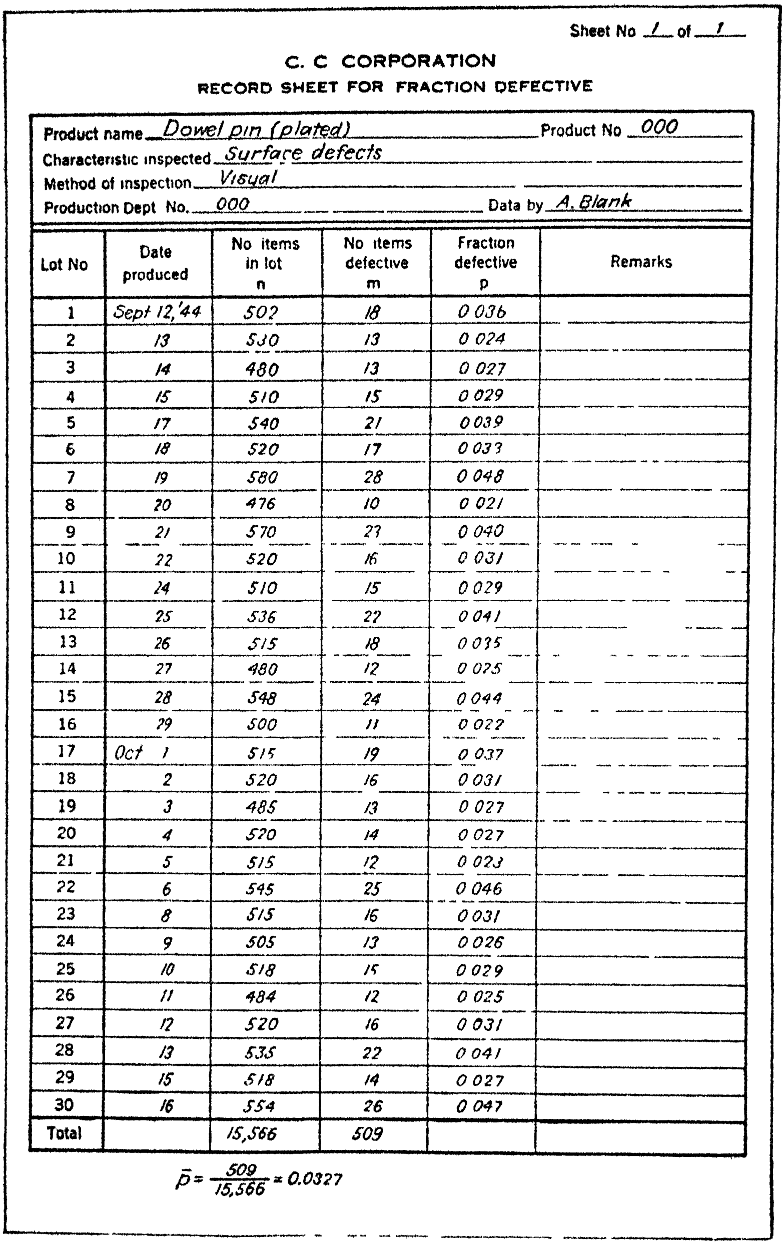
31. Forms for *p*-, *m*-, and *c*-Charts

It will be necessary to select or design a different form for use in recording data for the new charts than was used with the $\bar{\bar{X}}$- and *R*-charts. With each of the charts for fraction defective, for number of defectives, and for number of defects, we shall use a form like the one on which the data of Table 32 are recorded. Each chart will be drawn on cross-section paper, with the horizontal scale denoting the lot number and the vertical scale an appropriate one for the particular chart.

32. Chart for Fraction Defective

In general the procedure followed for constructing and using a control chart for fraction defective is quite similar to that for an $\bar{\bar{X}}$-chart or an *R*-chart. With a *p*-chart, as with the others, we draw a chart with a center line and control limits, but we have new formulas for computing control limits. It will be seen below that these formulas depend only upon $\bar{p}$, the average fraction defective, and $\bar{n}$, the average lot size.

TABLE 32



Sheet No 1 of 1

C. C CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Dowel pin (plated) Product No 000

Characteristic inspected Surface defects

Method of inspection Visual

Production Dept No. 000 Data by A. Blank

Lot No	Date produced	No items in lot n	No items defective m	Fraction defective p	Remarks
1	Sept 12,'44	502	18	0.036	
2	13	530	13	0.024	
3	14	480	13	0.027	
4	15	510	15	0.029	
5	17	540	21	0.039	
6	18	520	17	0.033	
7	19	580	28	0.048	
8	20	476	10	0.021	
9	21	570	23	0.040	
10	22	520	16	0.031	
11	24	510	15	0.029	
12	25	536	22	0.041	
13	26	515	18	0.035	
14	27	480	12	0.025	
15	28	548	24	0.044	
16	29	500	11	0.022	
17	Oct 1	515	19	0.037	
18	2	520	16	0.031	
19	3	485	13	0.027	
20	4	520	14	0.027	
21	5	515	12	0.023	
22	6	545	25	0.046	
23	8	515	16	0.031	
24	9	505	13	0.026	
25	10	518	15	0.029	
26	11	484	12	0.025	
27	12	520	16	0.031	
28	13	535	22	0.041	
29	15	518	14	0.027	
30	16	554	26	0.047	
Total		15,566	509		

$$\bar{p} = \frac{509}{15,566} = 0.0327$$

The procedure for constructing a p-chart will be given by means of an illustration, which will be supplemented by general explanations when needed. For this illustration we shall use data which resulted from visual inspection of plated pins for surface defects. The pins were inspected in lots of approximately 500, and each lot represented the total number inspected in a day. When the inspection for a day was completed, an entry was made in the appropriate column of Table 32 for each of the important records of the day, that is for the total number n of pins inspected, the total number m found defective, and the computed value of the fraction defective for the lot. As soon as the fraction defective had been computed and recorded in the table, the corresponding point was plotted on the chart of Fig. 32. On this chart the vertical scale is given as a per cent defective.

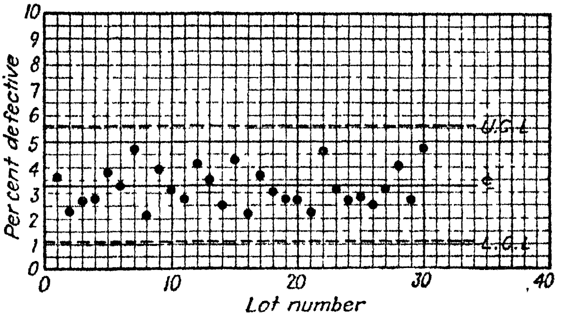


FIG 32.

The data for the 30 lots, recorded in Table 32, will be regarded as preliminary data from which to determine the condition of control of the production stream from which the lots were received. The average value $\bar{p}$ of the fraction defectives for the 30 lots will be taken tentatively to represent the normal expected fraction defective. Since in this illustration the lot size is not uniform the value of $\bar{p}$ should not be computed by averaging the values of p in the fifth column, but should be found by dividing the total number of defectives in the fourth column by the total number of pins in the third column. Hence we find

$$\bar{p} = \frac{509}{15{,}566} = 0.0327 = 0.033 \tag{1}$$

to three decimals. This value of $\bar{p}$ is used for locating the center line, as shown on Fig. 32.

The average lot size of 518.87, found by dividing the total number of pins 15,566 of the third column by 30, may be rounded off and taken as

$$\bar{n} = 520 \tag{2}$$

The general formula for computing the distance from the center line to the upper control limit is

$$Y_P = 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}} \tag{3}$$

Hence the location of the upper control limit is given by

$$\bar{p} + Y_P \tag{4}$$

If Y_P, as found by using (3), is less than $\bar{p}$ then the lower control limit is found by subtracting Y_P from $\bar{p}$. But if Y_P is greater than $\bar{p}$, so that $\bar{p} - Y_P$ would be negative, then the lower limit is taken as zero. Thus for the location of the lower limit we have

$$\begin{Bmatrix} \bar{p} - Y_P, & \text{if } Y_P < \bar{p} \\ 0, & \text{if } Y_P \geqq \bar{p} \end{Bmatrix} \tag{5}$$

For our illustration we substitute the values of $\bar{p}$ from (1), $\bar{n}$ from (2), and $1 - \bar{p}$ from (6),

$$1 - p = 1.000 - 0.033 = 0.967 \tag{6}$$

in (3) and find

$$Y_P = 3\sqrt{\frac{0.033 \times 0.967}{520}} = 3\sqrt{0.000061} = 3 \times 0.0078 \tag{7}$$

$$= 0.0234 = 0.023$$

to three decimals. Table I was used for finding the square root of 0.000061.

Substituting the value of p from (1) and of Y_P from (7) in (4) we have

$$\bar{p} + Y_P = 0.033 + 0.023 = 0.056 \tag{8}$$

for the location of the upper control limit. The same values substituted in (5) give

$$\bar{p} - Y_P = 0.033 - 0.023 = 0.010 \tag{9}$$

for the lower limit.

With the control limits from (8) and (9) located on the p-chart of Fig. 32 it is seen that all the points fall within the control limits and hence that the process from which the data of Table 32 were obtained was under control.

The two limits in (8) and (9) may be combined and written in the condensed form of

$$(10) \qquad \bar{p} \pm Y_P = 0.033 \pm 0.023 = \begin{cases} 0.056 & \text{U.C.L.} \\ 0.010 & \text{L.C.L.} \end{cases}$$

In any application of the condensed form given in (10), if the value of Y_P is greater than that of p, then the value for the lower limit would be written 0.

In general in an application of formula (3) the value of $\bar{p}$ used is the average fraction defective for all the lots considered, and the value of $\bar{n}$ for the average lot size is used provided that no lot size differs from $\bar{n}$ by more than 20 per cent of $\bar{n}$. The procedure to be followed when n differs from $\bar{n}$ by more than this will be considered in Art. 38.

33. Lower Limit Not Zero

The beginner is apt to think that the lower limit on a p-chart should always be zero. To him a point that approaches zero is approaching perfection and should call for no investigation. However, he should remember that all features of the chart, center line, control limits, and plotted points, result from the data obtained from the process. The formula for Y_P in (3), Art. 32, is a true formula from statistical methods, and the value obtained from it in any application gives the spread on either side of the center line within which points should fall that result from the same process that fixed Y_P, or that fixed $\bar{p}$ and $\bar{n}$. Of course it is not possible to have a negative fraction defective. Hence, if the process is so good that the value of $\bar{p}$ is small enough to call for a negative lower limit, then this limit must be taken as zero.

Since the formula for Y_P is given him without proof, the beginner might fail to have confidence in using it, and he might think that something is wrong with the formula when a lower limit determined by using it is not zero. Perhaps he could better appreciate the need for a lower limit that is different from zero if he would consider the desirability of so improving the process, when possible, that the average and the control limits would all be lowered. An investigation of the conditions which caused a point to fall below the lower limit might result in such an improvement.

34. Search for Trouble

Should lack of control be indicated by the chart for the preliminary data, then for each lot whose point falls outside of either control limit, an investigation should be made of conditions that prevailed when the lot was produced. If a point falls below the lower control limit, a search might reveal a condition that could be used to improve the process. In this case a correction should result in lowering the average and the control limits with subsequent production. On the other hand, should a point fall above the upper control limit a search might reveal some trouble that could be corrected. If the cause of trouble could be identified and a correction could be made that would remove the trouble from future production, then the data for all such points could be removed from the tabulations and new control limits computed from the remaining data. However, no data should be removed just because a point was outside a control limit, but only if some engineering reason was known for the unsatisfactory condition and a correction could be made that would prevent its occurrence in the future.

35. Standard Values for p-Chart

It must be decided whether or not the value of $\bar{p}$ as computed from 25 or 30 lots after the process has been brought under control is an acceptable normal value to be chosen as standard for the product. If this value of $\bar{p}$ represents a satisfactory condition of quality, as set by specifications, it may be adopted as standard and the control limits projected forward and used during future production. If the value of $\bar{p}$ is not acceptable then, as with the $\bar{X}$- and R-charts, it becomes necessary to decide whether to make changes in the process or to change specification requirements.

36. Lot Size

Usually each point on a p-chart represents the fraction defective, or per cent defective, for a fairly large lot, say 500 or more.

However, each lot size often corresponds with the number of items inspected in a day, and in such cases the lot size may vary, and may be much less than 500. The chart is not wholly satisfactory if there are frequent and wide variations in the lot size, or if the lot size is too small.

By use of the expression for Y_P in (3), Art. 32, it can be shown that if the lot size does not vary more than 20 per cent above or below the average n then this average lot size may be used for computing control limits. However, in a case such as this if a point falls at or near a control limit, either inside or outside the limit, it will be wise to carefully check the condition of control for the corresponding lot. If the point is well outside of a control limit then lack of control will be definitely indicated, but if the point is near a control limit it will be necessary to note whether the lot size n is greater or less than $\bar{n}$, and whether the point is inside or outside the control limit, before the condition of control for the lot can be determined. In some cases it may be necessary to compute the control limits for the lot in question and then see on which side of the recomputed limit the point falls.

Let us suppose, for example, that control limits have been computed based on the average $\bar{n}$ for 30 lots and that a few points fall near a control limit. If a point is just outside of a limit, lack of control is indicated if n for the lot is greater than $\bar{n}$, but if n is less than n then control limits must be computed, using n in place of n in formula (3), Art. 32, in order to determine the condition of control for the lot. On the other hand if a point is inside a control limit and n is less than $\bar{n}$ then control for the lot is established, but if n is greater than n then new control limits are needed for this lot. In either of these cases if the point falls outside the new control limit then the process was out of control when the corresponding lot was made.

The truth of the above statement will be revealed if we note that the average lot size $\bar{n}$ appears in the denominator of the fraction which is under the radical in (3), Art. 32. If in this formula a value of n that is larger than $\bar{n}$ is substituted for $\bar{n}$, the resulting value of Y_P will be smaller than that obtained by using $\bar{n}$, and hence the new control limit will be nearer the center line. The reverse of this is true if the substituted value of n is smaller than $\bar{n}$.

In Art. 38 attention will be given to the procedure to follow when the lot size varies frequently and by more than 20 per cent of $\bar{n}$. For the illustration given there, several applications will be made of recomputed control limits.

37. Use of Table II

In Article 32 we learned how to compute the spread from the center line to the control limits for a p-chart by use of (3) of that article, and we saw that an application of the formula required several operations of arithmetic. While these operations of arithmetic do not entail an excessive amount of labor, such labor as is required can often be reduced considerably by the use of Table II, page 157. Since we shall have numerous applications for which the spread will have to be found we shall now explain the use of the table.

In Table II numerous values of the lot size n, or the average lot size $\bar{n}$, are recorded in the first column under $\bar{n}$, and several values of fraction defectives are given in the first row opposite $\bar{p}$. Corresponding to each lot size and to each fraction defective that have been included in the table the value of $Y_P = 3\sqrt{[\bar{p}(1 - p)]/n}$ has been computed and entered in the row opposite the lot size and in the column under the fraction defective. For example, the value of Y_P for a lot size of 600 and a fraction defective of 0.04 is located in the row containing 600 and the column containing 0.04, and is 0.0240.

The values of Y_P have been computed and recorded to four decimal places but are usually used to the nearest three decimals. They are given to four places to aid in interpolation for values of $\bar{n}$ and $\bar{p}$ that are not given in the table.

It should be noted that the values of $\bar{n}$ that have been selected are such that any lot size under 20,000 will either agree with, or be within 20 per cent of, at least one table entry for $\bar{n}$. It should also be noted that the values of $\bar{n}$ range from 1 to 20,000, although lot sizes under 50 or over 2,000 should arise very infrequently.

Let us now introduce illustrations to show how to find values of Y_P for values of $\bar{n}$ or p, or both, that are not given in the table.

For an application for which the lot size is not given in the table suppose that we want the value of the spread Y_P for a lot which has 470 items and for which the fraction defective is 0.02. For

$\bar{p} = 0.02$ and $\bar{n} = 400$ we have $Y_P = 0.0210$, and for $\bar{p} = 0.02$ and $\bar{n} = 600$ we have $Y_P = 0.0172$. (We used 400 and 600 for $\bar{n}$ because they are the table entries between which 470 falls.) If we were interested in a rough check for one lot whose point fell too close to a previously computed control limit then we could use either of these two values of Y_P, although the safer value would be the one (0.0172) which corresponds to the larger of the two lot sizes between which 470 (the size of the lot in question) falls. This would be the safer because it would put the control limit slightly nearer to the center line than it would be for the computed value of Y_P for a lot size of 470. Hence if there were lack of control it would show more definitely than would be the case for the control limit for the table entry for a lot of size 400.

If it is desired to obtain a closer approximation to the value of Y_P for the lot size of 470, it could be found by interpolation. However, this interpolation need not be computed to too great a degree of accuracy. In general, if the lot size differs from either of the two table values of $\bar{n}$ between which it falls, by less than one-fourth of the difference between these two values of $\bar{n}$, then use the value of Y_P for the n that is nearer the lot size; otherwise use the arithmetic average of the values of Y_P corresponding to these two values of $\bar{n}$. Thus, for the present illustration, had the lot size been between 400 and 450 we would have used 0.0210, the value of Y_P corresponding to 400, while if the lot size had been between 550 and 600 we would have used 0.0172, the value of Y_P for a lot size of 600. Since 470 lies between 450 and 550 we can use the arithmetic average of the values of Y_P corresponding to lot sizes of 400 and 600. The resulting value of Y_P should be so close to the actual value that the discrepancy between them would be negligible if each should be plotted on the chart. Thus for a lot size of 470 we would have

$$(1) \qquad Y_P = \frac{0.0210 + 0.0172}{2} = \frac{0.0382}{2} = 0.0191 = 0.019$$

to 3 decimals, while by using (3), Art. 32, we find

$$(2) \qquad Y_P = 3\sqrt{\frac{0.02 \times 0.98}{470}} = 3\sqrt{\frac{0.0196}{470}} = 3\sqrt{0.000041702}$$
$$= 3 \times 0.00646 = 0.01938 = 0.019$$

to 3 decimals.

In this case the value of Y_P in (1), found by using the table, and that in (2), found by use of (3), Art. 32, agree to 3 decimal places. We should not always expect such close agreement. However, excepting perhaps for very small values of $\bar{n}$, the agreement should be close enough so that the value obtained by using the table would be acceptable. In any application in which the table is used, if the recomputed control limits are located on the chart and are found to fall very close to the plotted point for the lot under consideration then it would be wise to check by actually computing the value of Y_P by the use of (3), Art. 32.

Next let us consider an application for which the fraction defective does not correspond to one given in the table. For this application we shall find the value of Y_P for a lot which has 1,200 items, and for which the fraction defective is 0.043. The table values of Y_P corresponding to a lot size of 1,200 and fraction defectives of 0.040 and 0.050 are 0.0170 and 0.0189, respectively. From these we can obtain the value of Y_P corresponding to a fraction defective of 0.043, with sufficient accuracy, by adding three-tenths of the difference between 0.0170 and 0.0189 to 0.0170. This difference is

$$0.0189 - 0.0170 = 0.0019 \tag{3}$$

and three-tenths of this difference is

$$(0.3)(0.0019) = 0.00057 \tag{4}$$

Hence the value of Y_P for $\bar{n} = 1{,}200$ and $\bar{p} = 0.043$ is

$$\begin{aligned} Y_P &= 0.01700 + 0.00057 = 0.01757 \\ &= 0.0176 \text{ to 4 decimals} \qquad (5) \\ &= 0.018 \end{aligned}$$

to 3 decimals.

The reader can see the general application from this illustration and should note that it is necessary to add only as many tenths of the difference as is represented by the second significant digit of the fraction defective. Thus for a fraction defective of 0.041 he would add one-tenth of the difference, for 0.048 he would add eight-tenths of the difference, etc.

The actual value of Y_P for $\bar{n} = 1{,}200$ and $\bar{p} = 0.043$ is

$$
\begin{aligned}
Y_P &= 3\sqrt{\frac{0.043 \times 0.957}{1{,}200}} = 3\sqrt{0.00003437} = 3(0.00586) \\
\text{(6)} \qquad &= 0.01758 \\
&= 0.0176 \text{ to 4 decimals} \\
&= 0.018
\end{aligned}
$$

to 3 decimals. Such close agreement between the computed value and the value obtained from the table by this method of interpolation, as seen in (6) and (5), is not unusual, but should be expected, except perhaps for very small values of $\bar{n}$.

If we want the value of Y_P for a lot for which neither the lot size nor the fraction defective agrees with table values, we should use the larger of the two values of $\bar{n}$ between which the lot size falls and interpolate for the fraction defective as was done in the immediately preceding paragraph. This should give an acceptable value of Y_P unless the resulting control limit should fall very close to the corresponding point on the chart. Should that happen then the value of Y_P should be computed by formula (3), Art. 32. However, after we have had some experience with the use of the table we can usually decide whether or not the use of the formula will be necessary.

In the next article it will be necessary to compute the values of Y_P for several different values of $\bar{n}$, but all with the same fraction defective $\bar{p}$. We shall use formula (3), Art. 32, for the computations and shall introduce a method for simplifying the computations in successive applications of the formula. We shall check a few of the results by use of Table II.

38. Charts with Varying Values of *n*

The value of $\bar{p}$, the average fraction defective, is customarily taken as tentatively representative of the normal process from which all lots were taken, and is used for locating the center line for the preliminary stage, regardless of any variation in the lot size. If the number of items n for a lot does not differ from the average number of items $\bar{n}$ by more than 20 per cent of $\bar{n}$ then the control of quality for the lot is judged by whether or not the

TABLE 38

Sheet No. 1 of 1

C. C. CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name ________ Product No. ________
Characteristic inspected ________
Method of inspection ________
Production Dept. No. ________ Data by ________

Lot No.	Date produced	No. items in lot n	No. items defective m	Fraction defective p	Remarks
1		1512	56	0.037	
2		1620	65	0.040	
3		1350	49	0.036	
4		1440	55	0.038	
5		1812	71	0.039	
6		1680	67	0.040	
7		1754	65	0.037	
8		2420	90	0.037	
9		2520	111	0.044	
10		2382	114	0.048	
11		2120	51	0.024	
12		504	29	0.057	
13		1812	87	0.048	
14		1614	63	0.039	
15		1716	48	0.028	
16		1246	66	0.053	
17		1187	44	0.037	
18		1218	28	0.023	
19		1045	23	0.022	
20		928	51	0.055	
21		1127	44	0.039	
22		1032	37	0.036	
23		3024	103	0.034	
24		3160	76	0.024	
25		3520	102	0.029	
26		3311	106	0.032	
27		3240	149	0.046	
28		1590	75	0.047	
29		1480	40	0.027	
30		1606	53	0.033	
Total		54,970	2018		

$\bar{p} = \frac{2018}{54{,}970} = 0.037 \qquad \bar{n} = \frac{54{,}970}{30} = 1832$

fraction defective p for the lot when plotted on the chart falls within the control limits based on the expression $3\sqrt{[\bar{p}(1-\bar{p})]/\bar{n}}$. However, for any lot for which n differs from $\bar{n}$ by more than this allowed amount the control of quality must be judged by the spread from $\bar{p}$ obtained by computing Y_p for that lot using the expression $3\sqrt{[\bar{p}(1-\bar{p})]/n}$. Should two or more lots have lot sizes that differ from $\bar{n}$ by more than 20 per cent, but that do not differ from each other by so large a percentage, then the quality of these lots may be checked by using their average lot size.

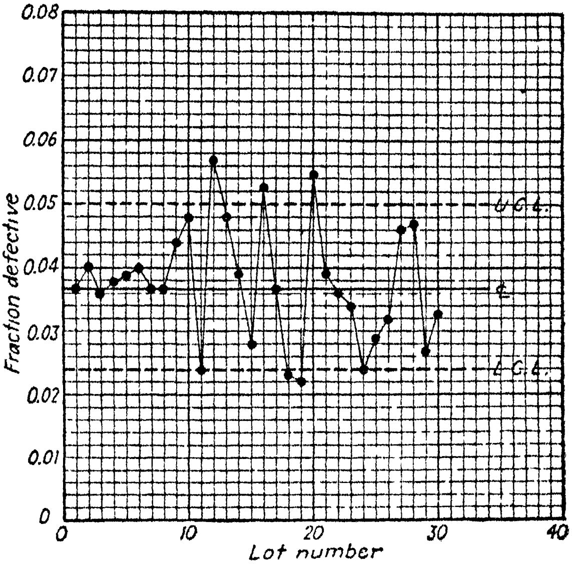


FIG. 38a.

We shall illustrate an application for varying lot size by using the assumed data that are given in Table 38. Many similar charts result from applications in practice, and often with more frequent variations in the lot size than is found in this illustration.

Since the number of items and the number of defectives represent assumed values that were chosen to illustrate certain conditions, the reader may assume any product, characteristic, specification requirements, and method of inspection that he wishes. He should try to visualize the illustration as an actual application, preferably one in his own department. He should think of each lot as being inspected, the number of items and the number of defectives recorded in Table 38, the fraction defective computed and recorded, and the corresponding point plotted on the chart of Fig. 38*a*, consecutively through the 30 lots.

As soon as Table 38 is completed and the 30 points are plotted on Fig. 38*a*, we must make the computations for the control limits and locate them on the chart.

Substituting

$$(1)\qquad \bar{n} = \frac{54{,}970}{30} = 1{,}832$$

$$(2)\qquad \bar{p} = \frac{2{,}018}{54{,}970} = 0.037$$

and

$$(3)\qquad 1 - \bar{p} = 1.000 - 0.037 = 0.963$$

in (3), Art. 32, we find the spread from the center line to the control limits to be

$$(4)\qquad Y_P = 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}} = 3\sqrt{\frac{0.037 \times 0.963}{1\ 832}} = 0.013$$

to 3 decimals.

To obtain Y_P by use of Table II we use $\bar{n} = 2{,}000$, the table entry next larger than 1,832. From the table we find for $\bar{n} = 2{,}000$, that for $\bar{p} = 0.030$, $Y_P = 0.0114$, and for $\bar{p} = 0.040$, $Y_P = 0.0131$. The difference between these two values of Y_P is

$$(5)\qquad 0.0131 - 0.0114 = 0.0017$$

and seven-tenths of this difference is

$$(6)\qquad (0.7) \times (0.0017) = 0.00119$$

Hence

$$(7)\qquad Y_P = 0.01140 + 0.00119 = 0.01259 = 0.013$$

to 3 decimals.

The value of Y_P from (4) or (7) substituted in the form of (10), Art. 32, gives the locations of the control limits as

$$(8)\qquad \bar{p} \pm Y_P = 0.037 \pm 0.013 = \begin{cases} 0.050 & \text{U.C.L.} \\ 0.024 & \text{L.C.L.} \end{cases}$$

We use the value of $\bar{p}$ from (2), and those for the locations of the control limits as given in (8), and locate the center line and the control limits on the chart of Fig. 38*a*. Then on examining the chart we note the following facts about points for various lots:

a. points for lots 11 and 24 fall on the lower control limit;
b. points for lots 12, 16, 18, 19, 20 fall outside of a control limit;
c. points for lots 10, 13, 27, 28, 29 fall near a control limit.

We shall examine the data for the lots indicated in *a*, *b*, and *c*, and see how many may be out of control. For this examination we shall use a new figure, Fig. 38*b*, with points plotted as on Fig. 38*a*, but with different lines for control limits in different portions of the chart. New control limits will be plotted for such of the lots in *a*, *b*, and *c*, above, as are found necessary from the discussion below. With the remaining lots the old limits of Fig. 38*a* will be used.

Since it will be necessary to find the spread Y_P for different lot sizes n (but each with the same value 0.037 for $\bar{p}$), let us write the expression for Y_P, from (3), Art. 32, for this illustration and reduce it as in

$$
\begin{aligned}
Y_P &= 3\sqrt{\frac{0.037 \times 0.963}{n}} = \frac{3\sqrt{0.037 \times 0.963}}{\sqrt{n}} \\
&= \frac{3\sqrt{0.0356}}{\sqrt{n}} = \frac{3 \times 0.189}{\sqrt{n}} \qquad (9) \\
&= \frac{0.567}{\sqrt{n}}
\end{aligned}
$$

Hence to compute the spread from the center line to the control limits for a lot (or a few lots) whose size differs from 1,832 by 20 per cent we have only to divide 0.567 by the square root of the lot size (or by the square root of the average of a few lots).

The number of items (2,382) in lot 10 exceeds the average by more than 20 per cent, and the number of items (2,120) in lot 11 is approximately 20 per cent more than the average. Since these lot sizes are larger than the average the control limit computed for either of them would be inside the limits for the chart of Fig. 38*a*, and, as either of the new limits might be inside of the corresponding plotted point, we should compute new control limits for the two points. As lots 8, 9, 10, and 11 are all large, but do not differ from each other by 20 per cent, let us compute control limits for these four points. The average size of these lots is

$$(10) \qquad \bar{n} = \frac{2{,}420 + 2{,}520 + 2{,}382 + 2{,}120}{4} = 2{,}360$$

Using the value of $\bar{n}$ from (10) in (9) we find

$$(11) \qquad Y_P = \frac{0.567}{\sqrt{2{,}360}} = \frac{0.567}{48.6} = 0.012$$

as the spread from the center line for points 8, 9, 10, 11. Hence we have, for the locations of their control limits

$$(12) \qquad \bar{p} \pm Y_P = 0.037 \pm 0.012 = \begin{cases} 0.049 & \text{U.C.L.} \\ 0.025 & \text{L.C.L.} \end{cases}$$

When these control limits are plotted on the chart, it is seen that point 10 is just inside the upper limit while point 11 is outside the lower one. Since the size of lot 11 differs from 2,360, the average size of lots 8, 9, 10, and 11, by almost as much as it does from 1,832, the control limit just obtained might not be suitable for lot 11. However, for lot 11 we have

$$(13) \qquad Y_P = \frac{0.567}{\sqrt{2{,}120}} = \frac{0.567}{46} = 0.012$$

in agreement with the value found in (11) as an average for lots 8, 9, 10, and 11. We can now say that the process was out of control when lot 11 was produced and that conditions which prevailed then should be investigated.

Point 12, which is quite far above the upper control limit on Fig. 38*a*, corresponds to a lot of 504 items, and as this is more than 70 per cent less than the average of 1,832 that was used in calculating the control limits, we must compute separate limits for this lot. Using $n = 504$ we find

$$(14) \qquad Y_P = \frac{0.567}{\sqrt{504}} = \frac{0.567}{22.4} = 0.025$$

With this value of Y_P we have

$$(15) \qquad \bar{p} \pm Y_P = 0.037 \pm 0.025 = \begin{cases} 0.062 & \text{U.C.L.} \\ 0.012 & \text{L.C.L.} \end{cases}$$

for the locations of the control limits for lot 12. When these limits have been drawn on the chart of Fig. 38*b* it is seen that the point for lot 12 is well inside of the upper limit, and hence we can decide that the process was under control when this lot was made.

Let us check this value by use of Table II. The value of $\bar{n}$ next larger than 504 is 600 and for this we find $Y_P = 0.0208$ for

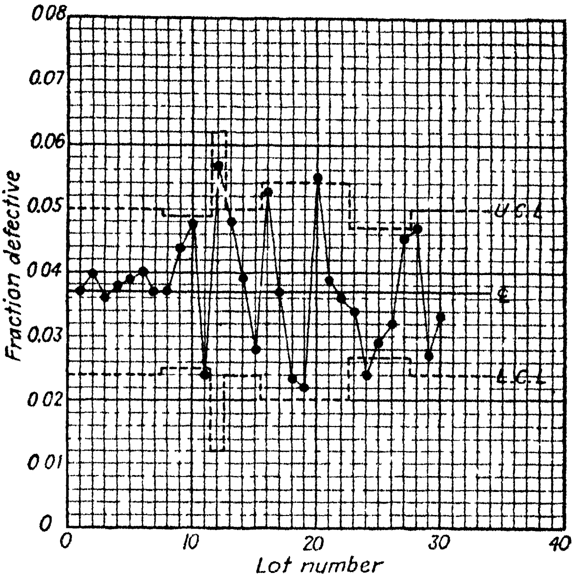


FIG. 38*b*.

$\bar{p} = 0.030$, and $Y_P = 0.0240$ for $p = 0.040$. The difference between these two values of Y_P is

$$0.0240 - 0.0208 = 0.0032 \tag{16}$$

and seven-tenths of this difference is

$$(0.7) \times (0.0032) = 0.00224 \tag{17}$$

Hence, by use of the table, we find for a lot size of 600 and a fraction defective of 0.037

$$Y_P = 0.0208 + 0.00224 = 0.02304 = 0.023 \tag{18}$$

This value of Y_P is slightly less than that found in (14) above. However, the difference is so slight that if we had determined the control limit by use of Y_P from (18) the point for lot 12 would be well inside of this limit. As we used the larger of the two lot sizes between which 504 (the number of items in lot 12) appears in the table, this control limit would be on the safe side and we could decide from the relative locations of the point and the control limit that the lot was made under controlled conditions.

The size of lot 13 (1,812 items) is so nearly equal to the average lot size (of 1,832) for all 30 lots that we should not make new computations for control limits. Hence, as the point is inside the old limits we accept lot 13 as under control.

Points 16, 18, 19, and 20, each of which falls outside of a control limit on Fig. 38*a*, represent lots that differ from the average for all 30 lots by more than 20 per cent. We shall compute new control limits for these points, and all the rest from 16 through 22, since their lot sizes do not differ from their own average by more than 20 per cent. For these we take

$$(19) \quad \bar{n} = \frac{1{,}246 + 1{,}187 + 1{,}218 + 1{,}045 + 928 + 1{,}127 + 1{,}032}{7} = \frac{7{,}783}{7} = 1{,}112$$

and find

$$(20) \qquad Y_P = \frac{0.567}{\sqrt{1{,}112}} = \frac{0.567}{33.3} = 0.017$$

Whence, we have

$$(21) \qquad \bar{p} \pm Y_P = 0.037 \pm 0.017 = \begin{cases} 0.054 & \text{U.C.L.} \\ 0.020 & \text{L.C.L.} \end{cases}$$

as the locations of the control limits for these points. When these limits are plotted on the chart of Fig. 38*b* we observe that points 16, 18, and 19 are inside the new control limits while point 20 is outside. Lots 16, 18, and 19 are in control but before we say that lot 20 is out of control we must consider its lot size. Since its size of 928 is almost 20 per cent under the average for these points as given in (19) we should compute new control limits for this lot. For this lot we have

$$(22) \qquad Y_P = \frac{0.567}{\sqrt{928}} = \frac{0.567}{30.5} = 0.019$$

whence the limits for point 20 are

$$(23) \qquad \bar{p} \pm Y_P = 0.037 \pm 0.019 = \begin{cases} 0.056 & \text{U.C.L.} \\ 0.018 & \text{L.C.L.} \end{cases}$$

From Fig. 38*b* it is seen that if the limits given in (23) were drawn point 20 would be just inside the upper one. As the point is so close to this limit and is outside the limit for the other points of approximately equal lot size, it would be wise to investigate conditions that prevailed when the lot was made.

Of the remaining points (24, 27, 28, 29), for which questions were raised in *a*, *b*, and *c* above, 28 and 29 correspond to lot sizes that are less than the average for all lots and since the points are inside the original control limits, these lots can be classified as in control without recalculating limits for them. As lots 24 and 27 appear with other lots from 23 through 27 whose sizes do not differ from their own average by as much as 20 per cent, let us compute limits for all of these lots. For these we have

$$(24) \qquad \bar{n} = \frac{3{,}024 + 3{,}160 + 3{,}520 + 3{,}311 + 3{,}240}{5}$$

$$= \frac{16{,}255}{5} = 3{,}251$$

whence

$$(25) \qquad Y_P = \frac{0.567}{\sqrt{3{,}251}} = \frac{0.567}{57} = 0.010$$

and the locations of the control limits for points 23 through 27 are given by

$$(26) \qquad \bar{p} \pm Y_P = 0.037 \pm 0.010 = \begin{cases} 0.047 & \text{U.C.L.} \\ 0.027 & \text{L.C.L.} \end{cases}$$

When these limits are plotted on the chart of Fig. 38*b* it is seen that point 24 is below the lower limit while point 27 is just inside the upper limit. The process was out of control when lot 24 was produced and an investigation should be made of conditions which prevailed then.

Figure 38*b* now shows the completed chart for the 30 lots.

39. Demonstration with Beads

The procedure for inspecting by attributes, and the results obtained, can be very closely duplicated by a demonstration using colored beads or marbles. The actual conditions of quality in a manufacturing process can be represented so accurately that it is possible to make numerous demonstrations that illustrate the procedure in constructing control charts for fraction defectives.

To make a demonstration we take a large number of beads of one color, say white, to represent good items, and a small number of another color, say red, to represent defective ones, and mix them together in a box or other container. The fraction defective for the total lot is the ratio of the number of red beads to the total of both colors.

If samples are taken and inspected 100 per cent, it will be found that some will have fraction defectives that are smaller, and others that are larger, than the fraction defective for the total lot. However, the average fraction defective for 25 or 30 samples should correspond very closely to the fraction defective for the lot from which samples are taken. This is similar to an application in a factory where the fraction defective for each day's run is found and the average of 25 or 30 of these should differ only slightly from the fraction defective for the items in the production stream from which the lots are taken. If, with the bead demonstration, points are plotted that correspond to fraction defectives of the samples, and control limits are calculated and located, the resulting chart will be a good representation of one in actual practice.

Let us give a brief description of equipment that is suitable for a demonstration and then we shall give an illustration using data obtained in a classroom demonstration. Ordinary colored beads, about 0.04 inch in diameter, that are used by children in kindergarten classes for making necklaces, are so easily obtained [1] that they are very satisfactory for this demonstration. It would be well to get 1,500 to 2,000 white beads and 300 to 500 red ones. The reader may wish to study questions of sampling, and he will find the beads also very useful for demonstrating certain conditions of sampling.

---

[1] Beads with diameters of 10 millimeters can be purchased from Walco Bead Co., New York, N. Y.

The beads of each color, in the quantities to represent the fraction for the demonstration, should be placed in a container in which they can be thoroughly mixed, and from which random samples can be taken by a paddle such as shown in either Fig. 39*b* or 39*c*. An ordinary shoebox makes a satisfactory container, although it is not very substantial. A more substantial box, which has given complete satisfaction, is shown in Fig. 39*a*.

The box does not have to meet any specification requirements and all dimensions can be changed by several inches if desired. The length could be reduced by 5 or 6 inches, and the depth increased by an inch or so. It would be well not to decrease the depth because of the danger of spilling beads out of the box when the paddle is lifted through them. The width should be sufficient to allow free motion of the paddle, but not enough to allow beads to get stuck between the edges of the paddle and the sides of the box.

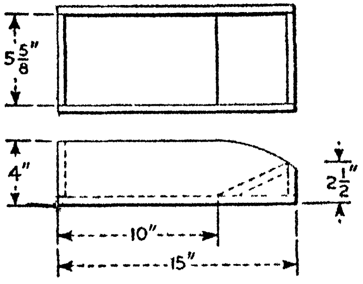


FIG. 39*a*.

The box can be constructed of plywood, plastic material, or any other durable material. The author has seen one which was made of a transparent plastic material, and which was used successfully with demonstrations for instructing men in a factory. The over-all length of the box was not more than 10 inches, and the depth was about 5 inches.

The paddle is shown with 50 holes so that 50 beads are selected with each sample. The paddle is usually made of hard wood. Sketches of two paddles, each of which has proven quite satisfactory, are shown in Figs. 39*b* and 39*c*.

Since the beads are not made under controlled conditions, the diameters will vary and the holes in the paddle should be just large enough to prevent the largest beads from getting stuck. If the holes are too large the smallest beads are apt to fall out of the hole as other beads roll down the face of the paddle. For a 10-millimeter bead a $^{13}/_{32}$-inch drill usually proves satisfactory. The holes should be about an eighth of an inch deep. With this depth of hole, if beads should get stuck a light tap of the face of the paddle on the edge of the box should remove them.

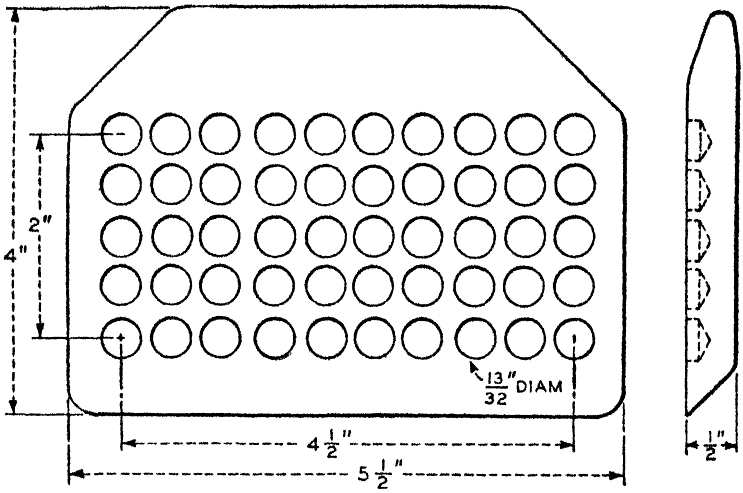


FIG. 39*b*.

There are, of course, some definite differences between conditions represented by the demonstration and those that actually occur with an application in a factory. These differences are

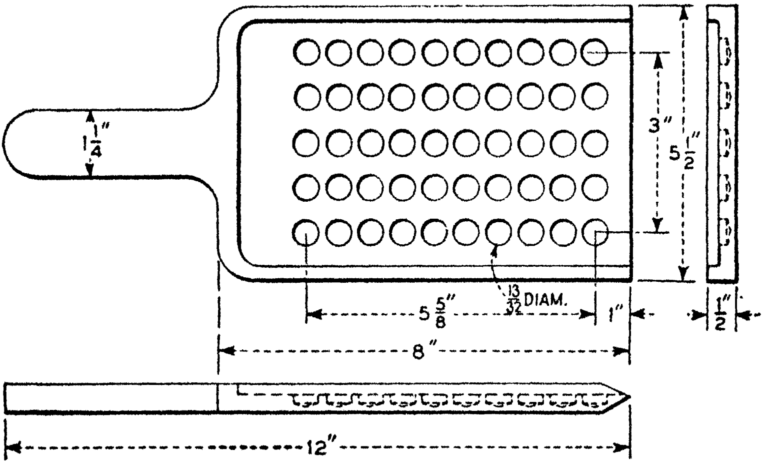


FIG. 39*c*.

similar to the ones noted with an $\bar{X}$-chart. However, if the reader can think of each sample as representing the production for one

TABLE 39

Sheet No. 1 of 2

C. C. CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Demonstration with beads Product No.

Characteristic inspected

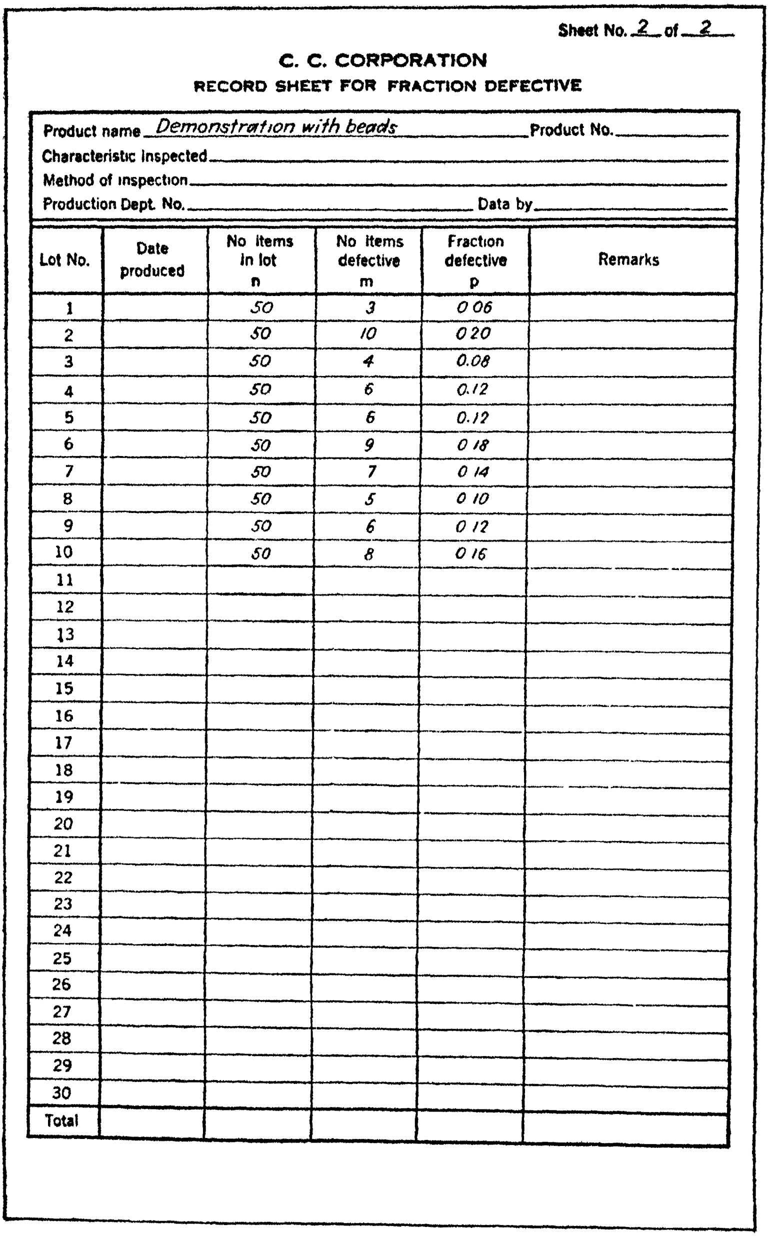
Method of inspection

Production Dept. No. Data by

Lot No.	Date produced	No. items in lot n	No items defective m	Fraction defective p	Remarks
1	Oct 21,'44	50	2	0.04	
2		50	3	0.06	
3		50	3	0.06	
4		50	5	0.10	
5		50	1	0.02	
6		50	1	0.02	
7		50	3	0.06	
8		50	2	0.04	
9		50	0	0.00	
10		50	1	0.02	
11		50	0	0.00	
12		50	3	0.06	
13		50	3	0.06	
14		50	3	0.06	
15		50	2	0.04	
16		50	0	0.00	
17		50	2	0.04	
18		50	1	0.02	
19		50	3	0.06	
20		50	0	0.00	
21		50	2	0.04	
22		50	4	0.08	
23		50	2	0.04	
24		50	1	0.02	
25		50	1	0.02	
26		60	2	0.04	
27		50	0	0.00	
28		50	8	0.16	
29		50	1	0.02	
30		50	3	0.06	
Total		1500	62		

$$\bar{p} = \frac{62}{1500} = 0.041$$

TABLE 39 (*continued*)



Sheet No. 2 of 2

C. C. CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Demonstration with beads Product No.

Characteristic inspected

Method of inspection

Production Dept. No. Data by

Lot No.	Date produced	No items in lot n	No items defective m	Fraction defective p	Remarks
1		50	3	0.06	
2		50	10	0.20	
3		50	4	0.08	
4		50	6	0.12	
5		50	6	0.12	
6		50	9	0.18	
7		50	7	0.14	
8		50	5	0.10	
9		50	6	0.12	
10		50	8	0.16	
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					
21					
22					
23					
24					
25					
26					
27					
28					
29					
30					
Total					

day, and further that each sample is taken from a continuous production stream and inspected 100 per cent, then the chart for fraction defective that results from the demonstration should give him a satisfactory picture of an application made under factory conditions.

Let us now give an illustration using the data of Table 39, which were obtained from a class demonstration.

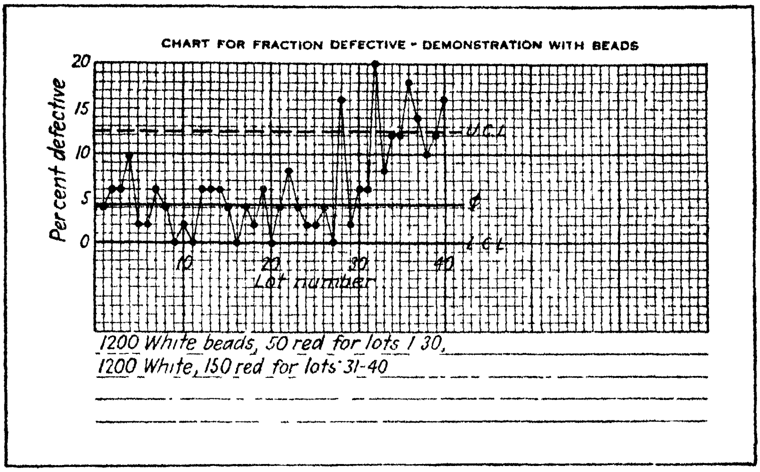


FIG. 39*d*.

There were 1,200 white and 50 red beads in the box, and the 1,250 beads were to represent a production stream with a fraction defective of

$$\bar{p} = \frac{50}{1,250} = 0.04 \tag{1}$$

Each paddle full of beads corresponded to one day's production and the number of red beads indicated the number of defective items in the lot. The number of red beads divided by 50 gave the fraction defective. With the class demonstration the number of defectives for each lot was recorded in the table, and the fraction defective was computed, recorded in the table and the corresponding point was plotted on the chart. After 30 lots had been chosen the center line and control limits were located on the graph. The preliminary chart then appeared as in Fig. 39*d* through point 30.

For the location of the center line we have

$$\bar{p} = \frac{62}{1,500} = 0.041 \tag{2}$$

Let us now use Table II for computing Y_P. With $\bar{n} = 50$ we find from the table that $Y_P = 0.0831$ for $\bar{p} = 0.040$, and $Y_P = 0.0925$ for $\bar{p} = 0.050$. The difference between these two values of Y_P is

$$0.0925 - 0.0831 = 0.0094 \tag{3}$$

and one-tenth of this difference is

$$(0.1) \times (0.0094) = 0.00094 \tag{4}$$

Thus we have

$$Y_P = 0.0831 + 0.00094 = 0.08404 = 0.084 \tag{5}$$

as the spread from the center line to the upper control limit. Since Y_P is greater than $\bar{p}$ the lower control limit will be taken as zero. Hence the control limits are

$$\bar{p} \pm Y_P = 0.041 \pm 0.084 = \begin{cases} 0.125 \\ 0.000 \end{cases} \tag{6}$$

When the upper control limit is drawn on Fig. 39*d* it is seen that all lots represented a satisfactory condition of control excepting the twenty-eighth. Of course in an actual application the conditions which prevailed when this lot was made would be investigated. Here it can be assumed as one of the 3 in 1,000 that could by chance fall outside of a control limit.

The demonstration has now been completed through the preliminary stage. The control limits found above and shown on the chart of Fig. 39*d*, should be suitable for adoption as standard, because we should expect only very slight changes to result if we should take a new set of samples and recompute $\bar{p}$. Therefore we shall proceed with the second phase of the demonstration.

We wish now to show how quickly a change in the control condition will show on the chart. If the process goes out of control, or if the level of control increases or decreases, the change should soon be revealed by the chart. If the normal average should increase the plotted points would have a general upward trend;

and if the increase should be appreciable, some points would fall outside of the upper control limit. Should the normal average decrease the trend would be downward, and should limits be recomputed the center line and the upper control limit would both be lowered.

We can cause an increase or a decrease in the normal average by adding more red beads to those in the box or by removing some from those already there. If we should add just one bead it would cause a slight increase in the fraction defective and this increase would eventually become evident. However, for such a slight change in the fraction defective it might take 20 or 25 samples to establish the new average, and there would be very little chance of a point going outside of the control limits previously determined. With 25, 50, 80, or any appreciable increment of red beads, we should have points go outside of the old control limits very soon after the new beads were placed in the box.

The data for sheet 2 of Table 39, and for points 31 to 40 of the chart on Fig. 39*d*, were obtained by continuing the demonstration after adding 100 red beads. The lot then in the box represented a controlled condition with a fraction defective of 0.111. Since the level of control was much higher than it was for the first 30 points, we should find, as we do, that the points on the chart are generally higher, with a large proportion of them falling above the old upper control limit.

Of course we should not compute new control limits for only 10 new lots. But in a practical application if the process should undergo as much of a change as was represented by adding 100 beads we should not have to plot as many as 10 points before the changed condition became evident. It should be noted that on Fig. 39*d*, point 31 is no higher than many in the first 30, but that 32 is above the upper control limit and that all the rest of the points are either above or not far below this limit. The average has quite evidently shifted upward.

Were this an application in a factory, an investigation would be started as soon as point 32 went out of control, and the process would be stopped after point 36, since it is out of control and 34 and 35 are so close to the control limit. The process would be suspended until the cause of trouble had been located. However, the demonstration was continued until 10 new lots had been selected in order to show that some points would fall outside of control and others would fall inside, and that there had been an obvious increase in the fraction defective or decrease in quality. The average fraction defective for these 10 lots is 0.128. Had we continued the demonstration until 30 new lots had been selected and had then computed the fraction defective for this second group of 30 lots, it would undoubtedly have been approximately equal to the average of 0.111 for the beads then in the box.

40. Chart for Number of Defectives

The procedure for construction of an m-chart, and the computations involved, are so similar to those for a p-chart that only brief explanations and comments will be needed in order to introduce these new charts.

Obviously it is a simple matter to construct a chart with the plotted points corresponding to the number of defective items in the lots. If each lot contains the same number of items then the condition of quality for the various lots can be readily interpreted by the locations of the plotted points. The center line would correspond to the average number of defectives, and this average should give a satisfactory indication of the quality of the product. Of course, since the average number of defectives $\bar{m}$ does not depend upon the lot size, but only upon the total number of defectives and the total number of lots, it is necessary to think of the number of defective items in terms of the size of the lots in order to get a true indication of the quality. This is, of course, equivalent to thinking in terms of the fraction defective. However, if you are making an m-chart you will soon learn to interpret the quality of a lot from the number of its defectives, or from the location of its point on the chart.

In connection with formulas for locating center line and control limits on a chart for number of defectives we shall use the following letters with the meanings indicated.

n = number of items in a lot
N = total number of lots on which averages are based
m = number of defective items in a lot
$\bar{m}$ = average number of defective items in N lots

The number of defective items in a lot is usually denoted by pn but we shall use m in order to have the number of defectives represented by a single letter. We can readily show that m and pn are equal because, by the definition of fraction defective, we have

$$(1) \qquad p = \frac{\text{number of defective items in a lot}}{\text{total number of items in the lot}} = \frac{m}{n}$$

and on multiplying the first and last members of equation (1) by n we obtain

$$(2) \qquad pn = m$$

As stated above, the center line for a chart for number of defectives corresponds to the average value of m for all the lots. Hence we have

$$(3) \qquad \bar{m} = \frac{\text{total number of defectives in all lots}}{\text{total number of lots}} = \frac{\Sigma m}{N}$$

where Σm represents the sum of the values of m for all the lots.

The spread from center line to control limits is given by

$$(4) \qquad Y_M = 3\sqrt{\bar{m}(1 - \bar{p})}$$

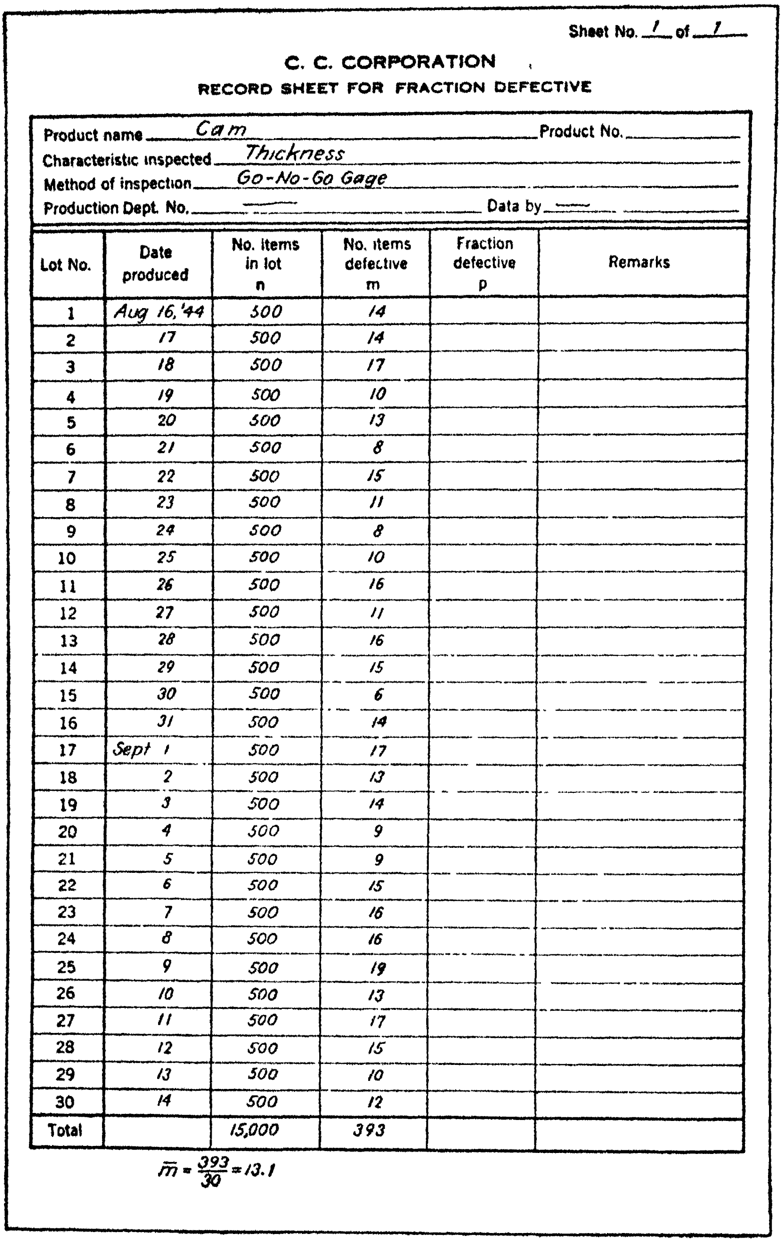
which is a formula that is obtained from courses on mathematical statistics.

In the large majority of applications the factor $1 - \bar{p}$, which appears under the radical in (4), is so nearly equal to unity that it is quite customary to take $1 - \bar{p} = 1$ and thus reduce (4) to

$$(5) \qquad Y_M = 3\sqrt{\bar{m}}$$

Formula (5) can be used as a satisfactory approximation to the spread from the center line to the control limits excepting in those cases where the fraction defective $\bar{p}$ is very large. Because of the ease with which computations can be made when formula (5) is used, this formula is recommended for use whenever possible, and certainly for fraction defectives less than 0.10. To use this formula for computing Y_M for any value of $\bar{m}$, it is necessary only to find the square root of $\bar{m}$ in Table I and to multiply this by 3.

TABLE 40



Sheet No. 1 of 1

C. C. CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Cam Product No.

Characteristic inspected Thickness

Method of inspection Go-No-Go Gage

Production Dept. No. —— Data by ——

Lot No.	Date produced	No. items in lot n	No. items defective m	Fraction defective p	Remarks
1	Aug 16, '44	500	14		
2	17	500	14		
3	18	500	17		
4	19	500	10		
5	20	500	13		
6	21	500	8		
7	22	500	15		
8	23	500	11		
9	24	500	8		
10	25	500	10		
11	26	500	16		
12	27	500	11		
13	28	500	16		
14	29	500	15		
15	30	500	6		
16	31	500	14		
17	Sept 1	500	17		
18	2	500	13		
19	3	500	14		
20	4	500	9		
21	5	500	9		
22	6	500	15		
23	7	500	16		
24	8	500	16		
25	9	500	19		
26	10	500	13		
27	11	500	17		
28	12	500	15		
29	13	500	10		
30	14	500	12		
Total		15,000	393		

$$\bar{m} = \frac{393}{30} = 13.1$$

Control limits as located with the value of Y_M from either (4) or (5) are

$$(6)\qquad \bar{m} \pm Y_M = \begin{cases} \bar{m} + Y_M & \text{U.C.L.} \\ \left.\begin{aligned} \bar{m} - Y_M, &\quad \text{if } \bar{m} > Y_M \\ 0, &\quad \text{if } \bar{m} \leqq Y_M \end{aligned}\right\} & \text{L.C.L.} \end{cases}$$

If we use Y_M from (5) then formula (6) may be expressed as

$$(7)\qquad \bar{m} \pm Y_M = \bar{m} \pm 3\sqrt{\bar{m}} = \begin{cases} m + 3\sqrt{m} & \text{U.C.L.} \\ \left.\begin{aligned} m - 3\sqrt{m}, &\ \text{if } \bar{m} > 9 \\ 0, &\ \text{if } \bar{m} \leqq 9 \end{aligned}\right\} & \text{L.C.L.} \end{cases}$$

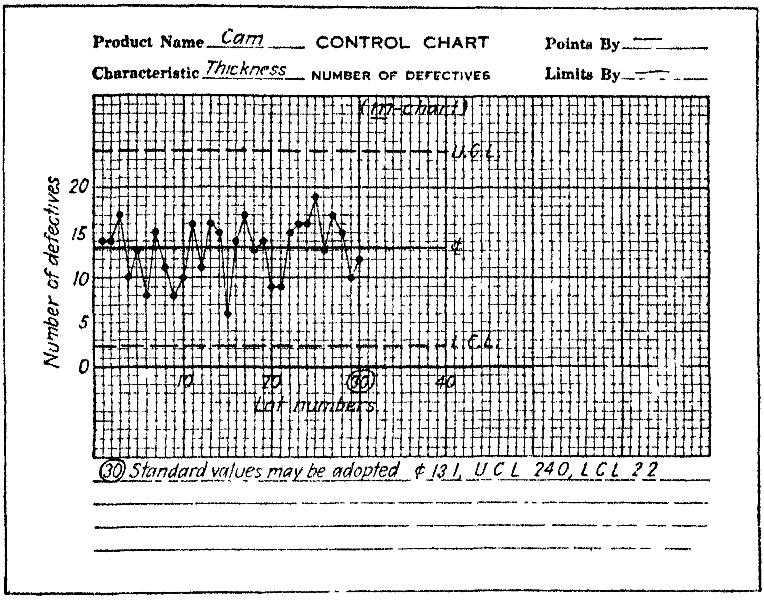


FIG. 40.

We shall now use the data of Table 40 as an illustration for an m-chart. These data are to represent the result of inspecting cams in lots of 500, where a go, no-go-gage was used to check each cam for thickness. As soon as a lot was inspected the appropriate entry was made in the table and the corresponding point was plotted on the chart of Fig. 40. When 30 lots had been inspected then the control limits were computed and located on the chart.

For the location of the center line we find

$$(8) \qquad \bar{m} = {}^{393}\!/_{30} = 13.1$$

The spread from the center line to the control limits is

$$(9) \qquad Y_M = 3\sqrt{\bar{m}} = 3\sqrt{13.1} = 3 \times 3.62 = 10.86 = 10.9$$

to one decimal. The square root of 13.1 was found by use of Table I. For the location of control limits we have

$$(10) \qquad \bar{m} \pm Y_M = \bar{m} \pm 3\sqrt{\bar{m}} = 13.1 \pm 10.9 = \begin{cases} 24.0 & \text{U.C.L.} \\ 2.2 & \text{L.C.L.} \end{cases}$$

When the center line and control limits are plotted on Fig. 40 it is seen that the points are all well inside the control limits. Since the control is satisfactory and the fraction defective,

$$(11) \qquad \bar{p} = \frac{393}{15{,}000} = 0.026$$

is small enough to be acceptable, the value of $\bar{m} = 13.1$ based on the first 30 lots may be accepted as standard, and the control limits may be projected forward for use with future production.

41. Chart for Number of Defects

In the definitions of Art. 30 it was stated that the number of defects per sample becomes useful in measuring quality if the possible number of defects per sample may be large compared with the average number. A sample may be a single article, as a designated length of wire, or a designated area of metal or strip of cloth, or sometimes it may be a designated number of items. Applications arise in inspecting metal plates, cloth, wire, etc., for finish castings for blowholes, an engine or a machine for number of assembly defects, and other inspections of which the reader can readily think.

The number of defects are usually represented by c, and consequently the chart is called a c-chart. The control limits will be dependent only upon the average number of defects $\bar{c}$.

The common practice is to interpret the procedure for a c-chart so that its development may be based on that for an m-chart. In fact until about 1942[1] the two charts carried the same designations. We shall use five steel plates to illustrate how a c-chart may be considered as a special case of an m-chart.

Suppose that we have five steel plates which have been carefully inspected and that we have found each plate has the number of pits or defects indicated in the upper left-hand corner of the plates shown on Fig. 41a.

Next let us suppose that we have examined the plate with five defects with detailed care and have discovered the pits located as on Fig. 41b. If we consider the plate as a lot with one unit then $n = 1$ and since that unit is defective the whole lot is defective and hence $p = 1$, or, in other words, the lot is 100 per cent defective. Obviously that would not be a satisfactory indication of quality.

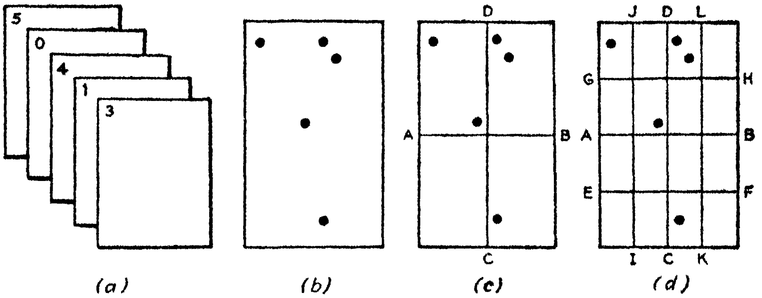


FIG. 41.

Instead of thinking of the plate as a lot which contains one unit we can divide the surface into four rectangles, by lines AB and CD as shown in Fig. 41c, and think of each small rectangle as a unit in a lot of 4 units. Then we would have $n = 4$, and as only 3 of the rectangles have defects the fraction defective will now appear as

$$p = 3/4$$

Let us think next of dividing the surface into 16 rectangles by adding lines EF, GH, IJ, and KL as shown in Fig. 41d. Among these 16 rectangles there are 4 that have defects and hence the fraction defective based on a lot of 16 items would be

---

[1] In Z1.2—1941, of the American Standards Association, both were identified by the designation pn. In Z1.3—1942 the designations used were pn for number of defectives and c for number of defects.

$$p = 4/16 = 1/4$$

If we should continue the process of subdividing the plate into smaller and smaller rectangles, we would eventually obtain a sufficiently large number of rectangles so that each pit would appear in a rectangle with no other pit.

In general we should now think of subdividing each plate in the above manner. We shall let n represent the largest number of rectangles that may be required on any one of the plates so that each pit would appear in a rectangle with no other pit, and shall think of each plate as subdivided into this same number n of rectangles. It will be unnecessary to actually determine the value of n, but we may think of it as a very large number that should be large enough to satisfy the conditions stated above.

We shall now think of these rectangles as n items, and of each rectangle that has a flaw as representing a defective item. On any plate the number of rectangles with flaws correspond to m, the number of defective items in a lot of n items. But this value of m is the same as the number of defects c on the surface, and we may now think of the fraction defective for a sample in terms of either m or c. Thus we have, for each plate,

$$(1) \qquad p = \frac{\text{number of rectangles with flaws}}{\text{total number of rectangles}} = \frac{m}{n}$$

$$(2) \qquad p = \frac{\text{number of defects in the sample}}{\text{a large number}} = \frac{c}{n}$$

In either (1) or (2) n is thought of as a large number, so that p will be small.

Following the form of (2) of the preceding article we may obtain from (1) and (2) above,

$$(3) \qquad pn = m = c$$

We are now ready to consider a chart for number of defects as a special application of a chart for number of defectives. If we should replace m by c the discussion of the preceding article would require but little alteration to apply to the present case.

The center line for a chart for number of defects corresponds to the average value of c for all the samples. Its value may be found by the formula

$$\bar{c} = \frac{\text{total number of defects for all samples}}{\text{total number of samples}} \tag{4}$$

Since the number of defects c per sample should be small and we may consider n as very large the value of $\bar{p}$, determined by the equation

$$\bar{p} = \frac{\bar{c}}{n} \tag{5}$$

will be small enough for us to consider $1 - \bar{p}$ as equal to 1. Hence we may take $1 - p = 1$ and use the form of (5) of the preceding article and compute the spread from the center line to the control limits by the formula

$$Y_C = 3\sqrt{\bar{c}} \tag{6}$$

Then for the location of control limits we have

$$\bar{c} \pm Y_C = \begin{cases} \bar{c} + 3\sqrt{\bar{c}} & \text{U.C.L.} \\ \left.\begin{array}{ll} \bar{c} - 3\sqrt{\bar{c}}, & \text{if } \bar{c} > 9 \\ 0, & \text{if } \bar{c} \lesseqgtr 9 \end{array}\right\} & \text{L.C.L.} \end{cases} \tag{7}$$

In any application of (6) the square root of $\bar{c}$ should be found by use of Table I.

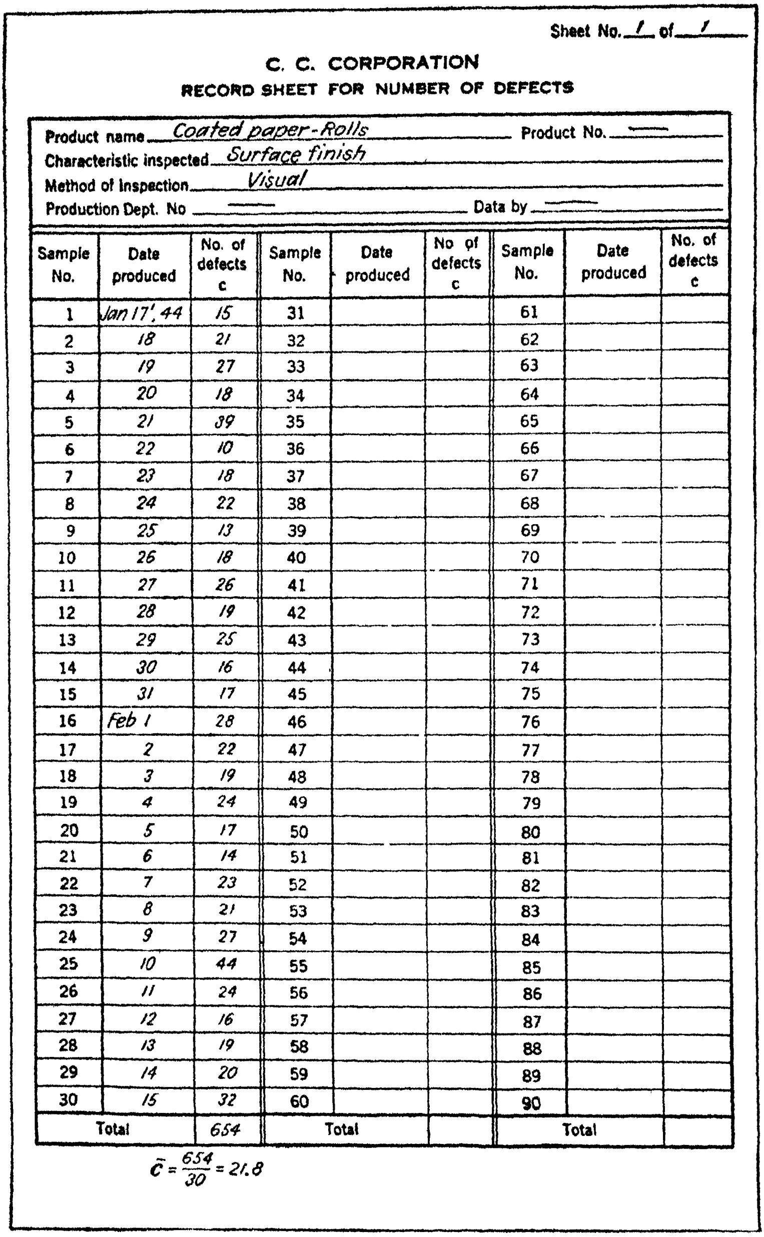
Let us apply the above discussion and formulas to our illustration with the steel plates. Of course we should not compute control limits on as few as 5 samples, but, as an example, let us do so for the 5 plates of Fig. 41*a*. With the values of c of 5, 0, 4, 1, and 3, as indicated on the plates of Fig. 41*a*, we find

$$\bar{c} = \frac{5 + 0 + 4 + 1 + 3}{5} = \frac{13}{5} = 2.6 \tag{8}$$

for the location of the center line. The spread from the center line to the control limits is

$$Y_C = 3\sqrt{\bar{c}} = 3\sqrt{2.6} = 4.8 \tag{9}$$

TABLE 41*a*



Sheet No. 1 of 1

C. C. CORPORATION

RECORD SHEET FOR NUMBER OF DEFECTS

Product name *Coated paper-Rolls* Product No. —

Characteristic inspected *Surface finish*

Method of inspection *Visual*

Production Dept. No — Data by —

Sample No.	Date produced	No. of defects c	Sample No.	Date produced	No of defects c	Sample No.	Date produced	No. of defects c
1	Jan 17, '44	15	31			61		
2	18	21	32			62		
3	19	27	33			63		
4	20	18	34			64		
5	21	39	35			65		
6	22	10	36			66		
7	23	18	37			67		
8	24	22	38			68		
9	25	13	39			69		
10	26	18	40			70		
11	27	26	41			71		
12	28	19	42			72		
13	29	25	43			73		
14	30	16	44			74		
15	31	17	45			75		
16	Feb 1	28	46			76		
17	2	22	47			77		
18	3	19	48			78		
19	4	24	49			79		
20	5	17	50			80		
21	6	14	51			81		
22	7	23	52			82		
23	8	21	53			83		
24	9	27	54			84		
25	10	44	55			85		
26	11	24	56			86		
27	12	16	57			87		
28	13	19	58			88		
29	14	20	59			89		
30	15	32	60			90		
Total		654	Total			Total		

$$\bar{c} = \frac{654}{30} = 21.8$$

Then for the location of the control limits we have

$$(10)\qquad \bar{c} \pm Y_C = \bar{c} \pm 3\sqrt{\bar{c}} = 2.6 \pm 4.8 = \begin{cases} 7.4 & \text{U.C.L.} \\ 0.0 & \text{L.C.L.} \end{cases}$$

Whence, if these limits had been based on a sufficient number of plates, we could say that a plate with eight pits would indicate lack of control.

We shall now give an illustration using the data of Table 41*a*. These data represent the number of defects found on inspecting rolls of coated paper. The rolls were approximately the same size and a reasonable number of defects per sample was around 20. After each roll was inspected and the number of defects was recorded in the table the corresponding point was plotted on the chart of Fig. 41*e*.

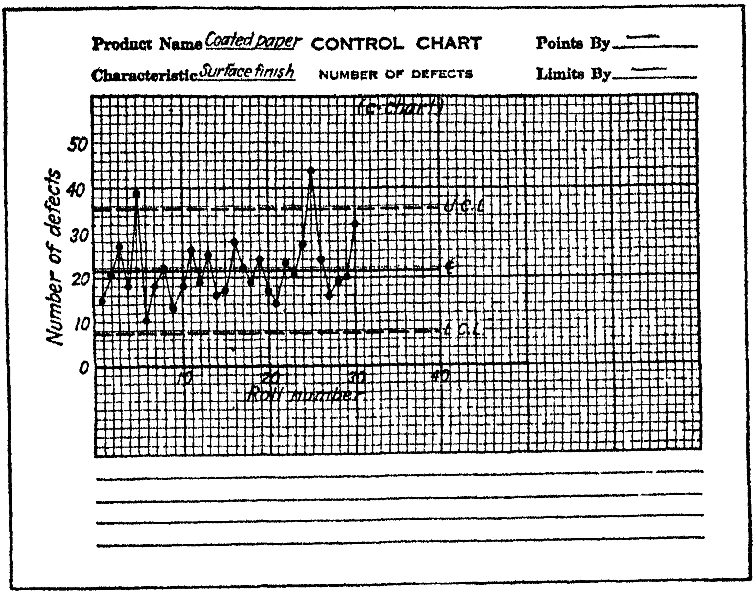


FIG. 41*e*

On this chart we have

$$(11)\qquad \bar{c} = {}^{654}\!/_{30} = 21.8$$

for the location of the center line, and

$$(12)\qquad Y_C = 3\sqrt{\bar{c}} = 3\sqrt{21.8} = 14.0$$

as the spread from the center line to control limits.

Using (11) and (12) we find

$$(13)\quad \bar{c} \pm Y_C = \bar{c} \pm 3\sqrt{\bar{c}} = 21.8 \pm 14.0 = \begin{cases} 35.8 & \text{U.C.L.} \\ 7.8 & \text{L.C.L.} \end{cases}$$

for the locations of the control limits. When the center line and control limits are drawn on the chart it is seen that all points are inside of control limits except 5 and 25. The conditions which prevailed when these two rolls were made should be investigated. If the cause for the trouble could not be identified then the control limits as found would be used as preliminary limits and search would be continued with future production to try and find the cause for trouble. However, if an assignable cause could be identified and corrective measures could be taken so that the condition would not affect future production, then we could recalculate the average and control limits based on the data remaining after omitting the data for these two from the table.

Let us suppose that an assignable cause had been found and corrections made and see what new control limits we would have. If we deduct 39 and 44, the number of defects on rolls 5 and 25, respectively, from the total of Table 41*a*, we then have 28 rolls with a total number of defects of 571. Hence we would obtain

$$\bar{c} = 571/28 = 20.4$$

for the new average, and

$$\bar{c} \pm 3\sqrt{\bar{c}} = 20.4 \pm 3\sqrt{20.4} = 20.4 \pm 13.5 = \begin{cases} 33.9 & \text{U.C.L.} \\ 6.9 & \text{L.C.L.} \end{cases}$$

for the control limits. These values would now be acceptable as standards.

EXERCISES

1. Make a chart for fraction defective (a p-chart) from data that you obtain by taking random lots of 50 beads from a box with 1,200 white and 50 red beads. Replace the beads in the box after each drawing and thoroughly mix before again obtaining a paddle full of 50 beads. With each lot count the number of red beads, enter this number in your record, compute the fraction defective, and plot the value of the fraction defective on your chart. After selecting 30 lots compute the average and the spread from center line to control limits, and locate the center line and control limits on the chart. Your average $\bar{p}$ should not differ very much from 0.04, which is represented by the beads in the box. Examine the data and the chart and determine whether or not a condition of control is indicated.

Add 25 red beads, thoroughly mix with the others in the box, and then draw at least 10 new lots of 50. Continue to draw lots of 50 until it becomes evident that the condition of control has changed and that the quality has decreased. Some, but not all, points should fall outside the upper control limit.

Add 25 more red beads, making 100 red and 1,200 white beads now in the box, and continue to draw lots of 50, to plot the points and to observe the change in quality.

Continue to add red beads and to make drawings of 50 until you have convinced yourself that a change in quality will be almost instantly reflected on the chart. You can, of course, reverse the process, removing some red beads and showing that an improvement in quality will also become evident from the chart.

2. Proceed as in Exercise 1 but starting with 1,500 white and 50 red beads.

3. The data of Table 41*b* represent results of inspecting engine cylinders, in lots of approximately 900.

Compute the fraction defective for each lot, the average fraction defective for the 30 lots, and the spread from the center line to the control limits.

Lay out the form for a chart for fraction defective (a *p*-chart), plot the fraction defective for each lot, and locate the center line and control limits.

Consider the question of control and discuss such conditions as may be revealed.

4. Follow the instructions of Exercise 3 with the data of Table 41*c* and obtain a chart for fraction defective (a *p*-chart) representing these data. In the discussion give particular attention to the variation in the number of items in the lots.

The data are to represent the record of inspecting plated gears for surface defects.

5. Make a chart for number of defectives (an *m*-chart) for the data of Table 41*d*. Locate the center line and the control limits and discuss the question of control.

The data are to represent the record of inspecting the threads on cap screws. The screws were inspected in lots of 500.

6. Make a chart for number of defectives (an *m*-chart) for the data of Table 41*e*. Locate the center line and the control limits and discuss the question of control.

The data are to represent the record of inspecting tank shoes for casting defects. The shoes were inspected in lots of 400.

7. Make a chart for number of defects (a *c*-chart) for the data of Table 41*f*. Locate the center line and the control limits and discuss the question of control.

The data are to represent records of inspecting 100-yard pieces of woolen goods for defects.

8. Make a chart for number of defects (a *c*-chart) for the data of Table 41*g*. Locate the center line and the control limits and discuss the question of control.

The data are to represent records of inspecting aircraft subassemblies for defects.

TABLE 41*b*

Sheet No 1 of 1

C C CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Cylinder Product No.

Characteristic inspected Thickness of walls

Method of inspection

Production Dept. No Data by

Lot No	Date produced	No items in lot n	No items defective m	Fraction defective p	Remarks
1	July 21,'44	902	36		
2	22	890	40		
3	23	908	58		
4	24	912	54		
5	25	905	38		
6	26	890	32		
7	27	907	29		
8	28	896	35		
9	29	906	37		
10	30	892	33		
11	31	887	29		
12	Aug 1	911	53		
13	2	909	57		
14	3	900	54		
15	4	910	31		
16	5	903	27		
17	6	891	24		
18	7	888	31		
19	8	894	34		
20	9	913	34		
21	10	900	43		
22	11	890	40		
23	12	908	30		
24	13	904	28		
25	14	906	29		
26	15	892	23		
27	16	896	26		
28	17	912	31		
29	18	910	41		
30	19	903	28		
Total					

TABLE 41c.

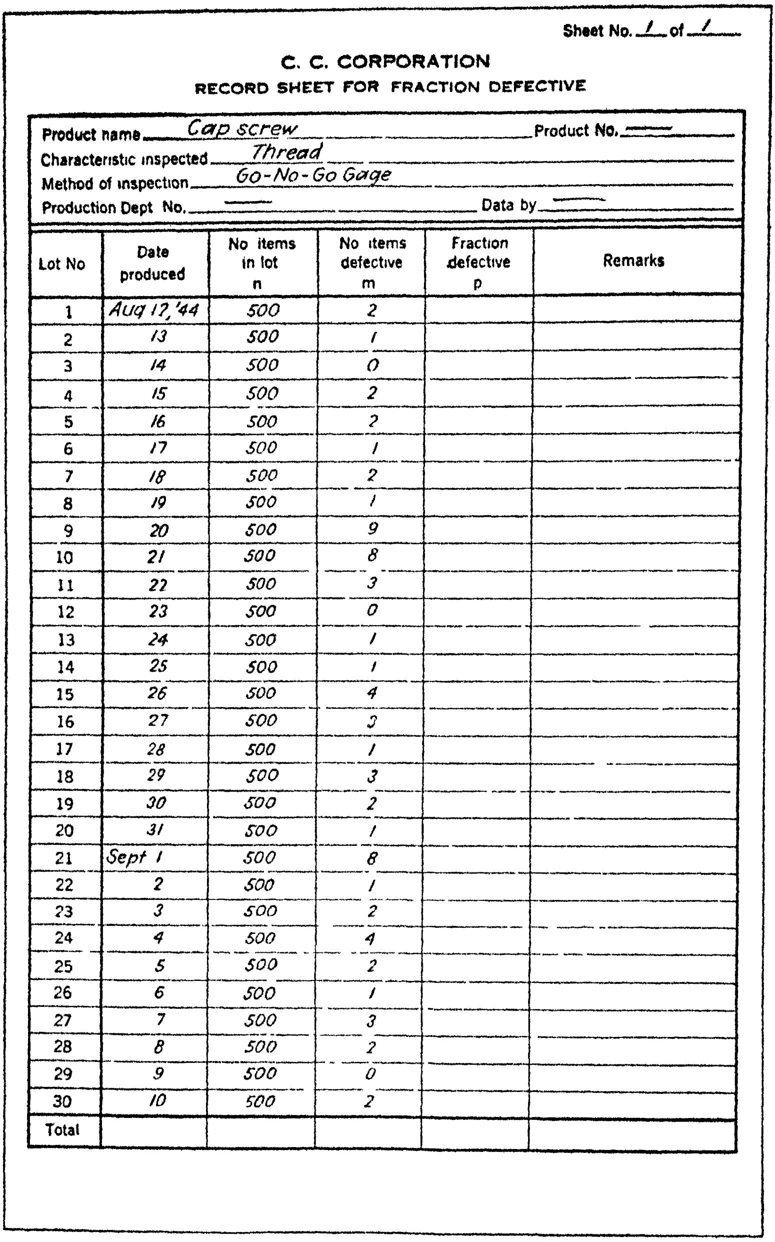
Sheet No. 1 of 1

C. C. CORPORATION
RECORD SHEET FOR FRACTION DEFECTIVE

Product name Plated gears Product No. —
Characteristic inspected Surface defects
Method of inspection Visual inspection
Production Dept. No. — Data by —

Lot No	Date produced	No. items in lot n	No. items defective m	Fraction defective p	Remarks
1	Sept 16, '44	2440	212		
2	17	2280	181		
3	18	2416	248		
4	19	1870	178		
5	20	1220	87		
6	21	1350	124		
7	22	1287	113		
8	23	1243	92		
9	24	1322	98		
10	25	1221	72		
11	26	1186	76		
12	27	1212	74		
13	28	1160	79		
14	29	1145	71		
15	30	1210	69		
16	Oct 1	1311	72		
17	2	1132	69		
18	3	1222	71		
19	4	1168	62		
20	5	1123	54		
21	6	1217	62		
22	7	1258	63		
23	8	1204	59		
24	9	1120	66		
25	10	1078	56		
26	11	1163	65		
27	12	862	54		
28	13	788	52		
29	14	834	44		
30	15	810	46		
Total					

TABLE 41*d*



Sheet No. 1 of 1

C. C. CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Cap screw Product No. ——

Characteristic inspected Thread

Method of inspection Go-No-Go Gage

Production Dept No. —— Data by ——

Lot No	Date produced	No items in lot n	No items defective m	Fraction defective p	Remarks
1	Aug 12,'44	500	2		
2	13	500	1		
3	14	500	0		
4	15	500	2		
5	16	500	2		
6	17	500	1		
7	18	500	2		
8	19	500	1		
9	20	500	9		
10	21	500	8		
11	22	500	3		
12	23	500	0		
13	24	500	1		
14	25	500	1		
15	26	500	4		
16	27	500	3		
17	28	500	1		
18	29	500	3		
19	30	500	2		
20	31	500	1		
21	Sept 1	500	8		
22	2	500	1		
23	3	500	2		
24	4	500	4		
25	5	500	2		
26	6	500	1		
27	7	500	3		
28	8	500	2		
29	9	500	0		
30	10	500	2		
Total					

TABLE 41e

Sheet No 1 of 1

C C CORPORATION

RECORD SHEET FOR FRACTION DEFECTIVE

Product name Tank shoes Product No. —
Characteristic inspected Casting defects
Method of inspection Visual
Production Dept No — Data by —

Lot No	Date produced	No items in lot n	No items defective m	Fraction defective p	Remarks
1	June 25 '44	400	12		
2	26	400	21		
3	27	400	9		
4	28	400	17		
5	29	400	18		
6	30	400	33		
7	July 1	400	15		
8	2	400	13		
9	3	400	3		
10	4	400	22		
11	5	400	11		
12	6	400	10		
13	7	400	7		
14	8	400	18		
15	9	400	32		
16	10	400	23		
17	11	400	19		
18	12	400	15		
19	13	400	24		
20	14	400	16		
21	15	400	14		
22	16	400	21		
23	17	400	34		
24	18	400	23		
25	19	400	14		
26	20	400	11		
27	21	400	8		
28	22	400	20		
29	23	400	12		
30	24	400	6		
Total					

TABLE 41*f*.

Sheet No. 1 of 1

C. C. CORPORATION

RECORD SHEET FOR NUMBER OF DEFECTS

Product name Woolen goods 100-yd. pieces Product No. —

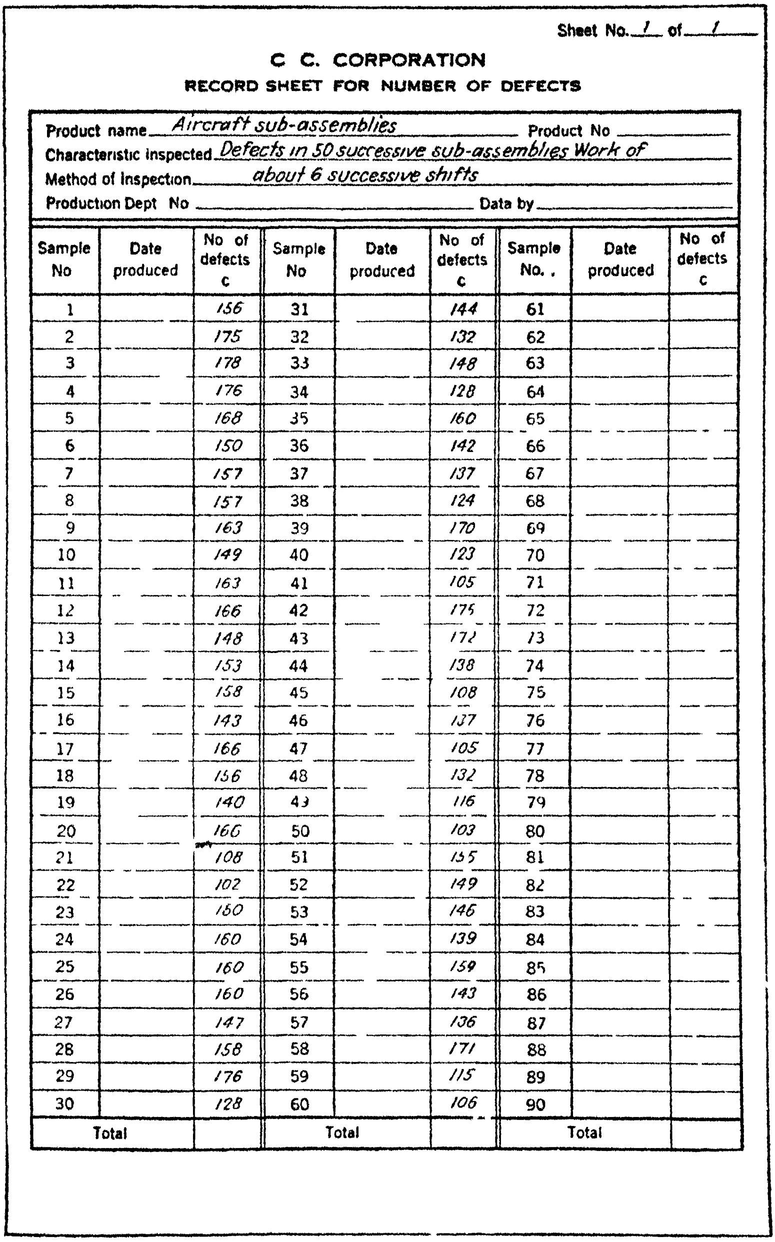
Characteristic inspected Defects

Method of inspection Visual

Production Dept. No. — Data by —

Sample No.	Date produced	No of defects c	Sample No.	Date produced	No. of defects c	Sample No.	Date produced	No of defects c
1	May, '44	2	31			61		
2		0	32			62		
3		0	33			63		
4		1	34			64		
5		1	35			65		
6		3	36			66		
7		1	37			67		
8		3	38			68		
9		0	39			69		
10		1	40			70		
11		3	41			71		
12		2	42			72		
13		5	43			73		
14		1	44			74		
15		4	45			75		
16		5	46			76		
17		0	47			77		
18		3	48			78		
19		1	49			79		
20		2	50			80		
21		1	51			81		
22		2	52			82		
23		3	53			83		
24		1	54			84		
25		1	55			85		
26		0	56			86		
27		0	57			87		
28		2	58			88		
29		1	59			89		
30		4	60			90		
Total			Total			Total		

TABLE 41*g*.



Sheet No. 1 of 1

C. C. CORPORATION
RECORD SHEET FOR NUMBER OF DEFECTS

Product name Aircraft sub-assemblies Product No

Characteristic inspected Defects in 50 successive sub-assemblies Work of

Method of inspection about 6 successive shifts

Production Dept No Data by

Sample No	Date produced	No of defects c	Sample No	Date produced	No of defects c	Sample No.	Date produced	No of defects c
1		156	31		144	61		
2		175	32		132	62		
3		178	33		148	63		
4		176	34		128	64		
5		168	35		160	65		
6		150	36		142	66		
7		157	37		137	67		
8		157	38		124	68		
9		163	39		170	69		
10		149	40		123	70		
11		163	41		105	71		
12		166	42		175	72		
13		148	43		172	73		
14		153	44		138	74		
15		158	45		108	75		
16		143	46		137	76		
17		166	47		105	77		
18		156	48		132	78		
19		140	49		116	79		
20		166	50		103	80		
21		108	51		155	81		
22		102	52		149	82		
23		150	53		146	83		
24		160	54		139	84		
25		160	55		159	85		
26		160	56		143	86		
27		147	57		136	87		
28		158	58		171	88		
29		176	59		115	89		
30		128	60		106	90		
Total			Total			Total		

CHAPTER V

THREE-SIGMA LIMITS

42. Relation to Standard Procedure

In Chaps. II through IV all of the charts commonly employed with statistical quality control were presented excepting the chart for standard deviation. While a few changes in form of presentation and of formulas from those of common practice were made, the resulting charts and the methods of interpretation conform in all essentials to those of standard practice. Where changes were made they were made in order to reduce the use of mathematical formulas and computations to a minimum. It was hoped that thereby some of the confusion that often accompanies the initial stages in the study of statistical quality control might be avoided.

Anyone who understands the material presented in the first four chapters, and can apply the methods and the formulas given there, should be able to make, to use, and to interpret charts for all applications that are likely to arise. However, he should now become familiar with all formulas of standard practice so that he can understand and can interpret charts made by others. Hence, in the present chapter all of the charts of statistical quality control will be considered and standard formulas will be presented.

It is hoped that the interest of many readers has been sufficiently aroused for them to want to proceed further in the study of theory and practice. Those who wish to proceed further should find that if they understand the standard formulas and the methods thus far presented, or presented in this chapter, they should be in a position to study any of the texts and to go as far as their inclination and their efforts will take them.

43. Standard Deviation

In Chap. II it was stated that frequency distributions and frequency curves play important roles in a statistical analysis. In that chapter we introduced limits for a frequency curve within which practically all items should fall that belong to the particular frequency distribution represented by the curve. It was stated that the limits, which were there called "danger limits," actually corresponded to important limits that were set by formulas of statistical methods. Then in Chap. III substitute formulas for those of statistical methods were used for locating danger limits on control charts.

In statistical methods these limits are based on a measure of the spread of the variables about their arithmetic average, or mean, and this measure is known as the *standard deviation* of the variables. Standard deviation is *the root-mean-square deviation from the arithmetic mean.* It is a measure of the dispersion of the variables on either side of the mean, or a measure of the tendency of the values of the variables to concentrate about the average. The closer these are to the average the smaller is the standard deviation.

Standard deviation is denoted by the lower-case Greek letter sigma (σ) and is computed by the formula

$$(1) \qquad \sigma = \sqrt{\frac{(X_1-\bar{X})^2+(X_2-\bar{X})^2+(X_3-\bar{X})^2+\cdots+(X_n-\bar{X})^2}{n}}$$

where X_1, X_2, X_3, $\cdots$, X_n are the n observed values of a frequency distribution and $\bar{X}$ is the arithmetic average, or mean. The average $\bar{X}$ is found by the formula

$$(2) \qquad \bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

The expression under the radical in (1) when reduced gives

$$(3) \qquad \sigma = \sqrt{\frac{X_1^2 + X_2^2 + X_3^2 + \cdots + X_n^2}{n} - \bar{X}^2}$$

as a simpler formula than (1) for computing the value of σ.

Let us take a very simple example and show how to compute σ from both formulas (1) and (3). Suppose that we have 5 values of X such as $X_1 = 6$, $X_2 = 7$, $X_3 = 10$, $X_4 = 11$, and $X_5 = 14$. Then using (2) we find

$$\bar{\bar{X}} = \frac{6 + 7 + 10 + 11 + 14}{5} = \frac{48}{5} = 9.6 \tag{4}$$

and from (1) we have

$$\begin{aligned} \sigma &= \sqrt{\frac{(6-9.6)^2+(7-9.6)^2+(10-9.6)^2+(11-9.6)^2+(14-9.6)^2}{5}} \\ &= \sqrt{\frac{(-3.6)^2 + (-2.6)^2 + (0.4)^2 + (1.4)^2 + (4.4)^2}{5}} \\ &= \sqrt{\frac{12.96 + 6.76 + 0.16 + 1.96 + 19.36}{5}} = \sqrt{\frac{41.20}{5}} = \sqrt{8.24} \\ &= 2.87 \end{aligned} \tag{5}$$

Using (3) we obtain

$$\begin{aligned} \sigma &= \sqrt{\frac{6^2 + 7^2 + 10^2 + 11^2 + 14^2}{5} - (9.6)^2} \\ &= \sqrt{\frac{36 + 49 + 100 + 121 + 196}{5} - 92.16} \\ &= \sqrt{\frac{502}{5} - 92.16} = \sqrt{100.4 - 92.16} = \sqrt{8.24} \\ &= 2.87 \end{aligned} \tag{6}$$

as before.

The computations in (5) and (6) can be facilitated by using the table of squares and square roots in Table I.

Sigma is used in courses on statistical methods as the most satisfactory measure of the dispersion of individual items of a frequency distribution about their arithmetic average. Three sigma gives the spread on either side of the average within which 99.73 per cent of the items must fall that are represented by the distribution, if the distribution is normal.

Let us now use one of the distributions of Chap. II as an illustration.

The frequency distribution in Table 43*a* is that of the outside diameters of 150 bushings that was given in Table 16*b*.

TABLE 43*a*.—FREQUENCY DISTRIBUTION FOR O.D. OF 150 BUSHINGS

Diameter, inches X	Frequency in 150 f
2.004	2
2.003	6
2.002	14
2.001	32
2.000	41
1.999	30
1.998	18
1.997	6
1.996	1
Total. . . .	150

It is unnecessary to actually identify the various table entries under X as X_1, X_2, etc. It should be obvious that, in order to use formula (3) for computing sigma, each entry under X must be squared and then multiplied by the corresponding frequency given under f. The computations and the results are readily found by use of the form given in Table 43*b*. In this table we have used 0, +1, +2, +3, +4, −1, −2, −3, and −4 to represent 2.000, 2.001, 2.002, 2.003, 2.004, 1.999, 1.998, 1.997, and 1.996, respectively.

TABLE 43*b*.—FORM FOR USE IN COMPUTING σ FOR O.D. OF 150 BUSHINGS

X	f	fX	fX^2
+4	2	8	32
+3	6	18	54
+2	14	28	56
+1	32	32	32
0	41	0	0
−1	30	−30	30
−2	18	−36	72
−3	6	−18	54
−4	1	−4	16
Total. .	150	−2	346

The entries in the column headed fX are obtained by multiplying together corresponding entries under X and f. This product is negative for negative values of X. The entries under fX^2 are obtained by multiplying together corresponding entries under X and under fX. These are all positive because when X is negative fX is also and the product is positive.

For the sake of ease of expression in formulas let us use the symbol of summation Σ, which was introduced in Art. 40. Thus

$$\Sigma f = 150 = n, \qquad \Sigma fX = -2, \qquad \text{and} \qquad \Sigma fX^2 = 346$$

We then have

$$\bar{X} = \frac{\Sigma fX}{n} = \frac{-2}{150} = -0.013 \tag{7}$$

in thousandths of an inch. The average $\bar{X}$ is thus seen to be 0.013 thousandths of an inch below the zero value. This is equivalent to 0.000013 inch below 2.0000, which was taken as zero in Table 43*b*. Hence, we have

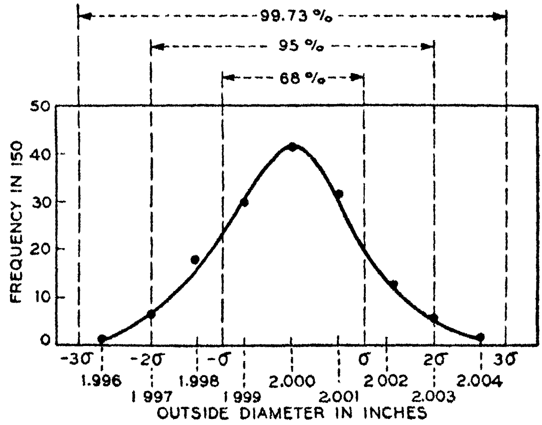


FIG. 43.

$$\bar{X} = 2.000000 - 0.000013 = 1.999987 = 2.0000 \tag{8}$$

to four decimals. As we wish accuracy to only one place more than was used in making the measurements we shall use 2.0000 as the average.

To compute sigma we use (3) and find

$$\sigma = \sqrt{\frac{\Sigma f X^2}{n} - \bar{X}^2} = \sqrt{\frac{346}{150} - (0.013)^2}$$

$$= \sqrt{2.3067 - 0.0002} = \sqrt{2.3065} \tag{9}$$

$$= 1.5 \text{ in thousandths of an inch}$$

$$= 0.0015 \text{ in inches}$$

In Fig. 43 the frequency curve for the data of Table 43*a* has been drawn, and intervals on the horizontal scale have been designated at distances σ, 2σ, and 3σ from the average, those to the right being taken as positive, those to the left as negative. These should be measured from the average $\bar{X}$, and not from the desired central value. In this illustration the two are identical so that there is no distinction between them. The locations of these several values of sigma are thus found from

$+\sigma$, $2.0000 + 0.0015 = 2.0015$

$-\sigma$, $2.0000 - 0.0015 = 1.9985$

$+2\sigma$, $2.0000 + 2 \times (0.0015) = 2.0030$

-2σ, $2.0000 - 2 \times (0.0015) = 1.9970$

$+3\sigma$, $2.0000 + 3 \times (0.0015) = 2.0045$

-3σ, $2.0000 - 3 \times (0.0015) = 1.9955$

In courses in statistical methods it is shown that the percentage distribution of a set of variables that are controlled by the same conditions, or that belong to the same frequency distribution, is approximately as indicated on Fig. 43. (Refer back to Fig. 24*a*.) Thus if there is no change in the process of manufacturing the bushings then 99.73 per cent of all that are made should fall within a range of $\pm 3\sigma$ from the average. In other words, only about 3 out of 1,000 should be made with an outside diameter less than 1.9955 or greater than 2.0045.

In like manner approximately 95 per cent of the variables fall within $\pm 2\sigma$ of the average, and 68 per cent between $\pm \sigma$. For the bushings approximately 95 per cent should fall between 1.9970 and 2.0030, and 68 per cent between 1.9985 and 2.0015.

The percentage distributions, as shown in Fig. 43 for the bushings, correspond to the general percentage distributions that are given in courses in statistical methods as applying to a controlled set of variables. Since practically all of the individual values of a controlled set of variables should fall within a spread of 3σ above or below the average value, the limits of this spread are called *three-sigma limits.* These limits are 3-σ limits for individual items.

Control limits on a control chart are based on, and are known as, 3-σ limits. Control limits are actually 3-σ limits of averages. That is, if the average value of each sample is treated as an individual item, and if 3-σ limits are computed for the averages of all of the samples, then these limits will correspond to the control limits for the items contained in all of the samples. If the average for a sample is plotted and the point falls within the control limits, then the individual items of the sample should fall within the 3-σ limits for the individual items of the total distribution.

The 3-σ limits for the averages of samples which give the control limits for the control charts, are customarily found by the formulas of Chap. III, using $\bar{R}$ and table values for the factor A_2. While we have referred to those formulas as "substitute formulas," they are standard formulas, and control limits found by their use will in general give a satisfactory approximation to those found by computations for 3-σ limits.

The reader should now recognize that the danger limits of Chaps. II and III were actually 3-σ limits. In fact the 3-σ limits found above for the bushings were given as danger limits in Chap. II when the frequency distribution of the bushings was being considered.

44. Comparison of Formulas

In this article we shall give a list of formulas that are customarily used in computing control limits. We shall compare the formulas of Chaps. III and IV with the standard formulas and show that the majority of the formulas that have thus far been used in this text are identical in final form with corresponding standard formulas. When there is a difference it is in form and not in final result. Hence the reader should have no trouble in adjusting his methods as developed here to conform to those of other texts.

The formulas and the notation will be those of the American War Standard Z1.3—1942. The values of A, A_1, A_2, B_3, B_4, D, D_3, D_4, d_2, etc., are given in Table III.

a. Chart for Averages, Using $\bar{R}$.—Formulas:

$$\text{℄},\quad \bar{\bar{X}} \tag{1}$$

$$\text{U.C.L.},\quad \bar{X} + A_2\bar{R} \tag{2}$$

$$\text{L.C.L.},\quad \bar{\bar{X}} - A_2\bar{R} \tag{3}$$

The center line (1) agrees with e 1, Art. 22, while formulas (2) and (3) agree with (2) and (3) as combined in (5), of Art. 22. In that article the values of A_2 were given for samples of 4 and 5. Values of A_2 for samples 2 to 10 are listed in Table III.

b. Chart for Averages, Using σ.—Formulas:

$$\text{℄},\quad \bar{X} \tag{4}$$

$$\text{U.C.L.},\quad \bar{X} + A_1\sigma \tag{5}$$

$$\text{L.C.L.},\quad \bar{\bar{X}} - A_1\bar{\sigma} \tag{6}$$

Charts for averages, based on σ, have not been previously given in this text. To obtain $\bar{\sigma}$ we first compute σ for each sample and then average all of these sigmas. The results from using formulas (5) and (6) are generally only slightly more accurate for determining control limits than are those from formulas (2) and (3). Since the latter require less labor in computations they are used more frequently than the former.

c. Chart for Ranges.—Formulas:

$$\text{℄},\quad \bar{R} \tag{7}$$

$$\text{U.C.L.},\quad D_4\bar{R} \tag{8}$$

$$\text{L.C.L.},\quad D_3R \tag{9}$$

Formula (7) gives the location of the center line as it was indicated in section h of Art. 22. Although formulas (8) and

(9), and (17) and (18), of Art. 22 differ in form, the locations obtained for the control limits by the two sets of formulas are the same. Formulas (8) and (9) give the distance up from zero to the control limits, while the corresponding formulas in Art. 22 give the spread above and below the center line $\bar{R}$. The resulting locations for the control limits are the same. That is,

$$\bar{R} + D\bar{R} = D_4\bar{R} \tag{10}$$

and

$$\bar{R} - D\bar{R} = D_3\bar{R} \tag{11}$$

Each side of equation (11) is zero if the sample size is 6 or less.

d. Chart for Standard Deviation.—Formulas:

$$\text{℄}, \quad \bar{\sigma} \tag{12}$$

$$\text{U.C.L.}, \quad B_4\bar{\sigma} \tag{13}$$

$$\text{L.C.L.}, \quad B_3\bar{\sigma} \tag{14}$$

The σ-chart was not presented in Chap. III; hence there are no formulas for comparison.

e. Chart for Fraction Defective.—Formulas:

$$\text{℄}, \quad p' \tag{15}$$

$$\text{U.C.L.}, \quad p' + 3\sqrt{\frac{p'(1 - p')}{n}} \tag{16}$$

$$\text{L.C.L.}, \quad p' - 3\sqrt{\frac{p'(1 - p')}{n}} \tag{17}$$

Formulas (1), (3), (4), and (5) of Art. 32, when used as they were in that article, give the same locations of center line and control limits as are found with (15), (16), and (17) above. In the formulas of Art. 32 $\bar{p}$ and n appeared in place of p' and n. In Art. 32 Table II was frequently used for finding the value of $3\sqrt{\frac{\bar{p}(1 - \bar{p})}{\bar{n}}}$, and the lower limit was taken as zero if $\bar{p}$ was not greater than $3\sqrt{\frac{\bar{p}(1 - p)}{\bar{n}}}$.

f. Chart for Number of Defectives.—Formulas:

$$\text{℄}, \quad p'n \tag{18}$$

$$\text{U.C.L.}, \quad \begin{cases} p'n + 3\sqrt{p'n(1 - p')} \\ p'n + 3\sqrt{p'n}, & \text{if } p' < 0.05 \end{cases} \tag{19}$$

$$\text{L.C.L.}, \quad \begin{cases} p'n - 3\sqrt{p'n(1 - p')} \\ p'n - 3\sqrt{p'n}, & \text{if } p' < 0.05 \end{cases} \tag{20}$$

The same formulas were given in Art. 32 with $\bar{m}$ in place of $p'n$. In that article the simplified form of the radical, $3\sqrt{\bar{m}}$, was used and the lower limit was taken as zero if $\bar{m} \leqq 9$. Table I was used for finding values of $3\sqrt{\bar{m}}$.

g. Chart for Number of Defects.—Formulas:

$$\text{℄}, \quad c' \tag{21}$$

$$\text{U.C.L.}, \quad c' + 3\sqrt{c'} \tag{22}$$

$$\text{L.C.L.}, \quad c' - 3\sqrt{c'} \tag{23}$$

The formulas for c-charts given in Art. 32, with $\bar{c}$ in place of c, are identical with corresponding ones in (21), (22), and (23). In that article Table I was used for finding the values of $3\sqrt{\bar{c}}$, and the lower limit was taken as zero if $\bar{c} \leqq 9$.

45. Standard Values

In the American War Standard Z1.3, and other texts, attention is given to the determination of standard values of the average, and of the control limits, for the control charts. In some cases these standard values are specified before the process begins, but standard values are usually determined and adopted after the process has been brought under satisfactory control.

In Z1.3 standard values of a variable are denoted by priming the letter that represents the variable. Thus $\bar{X}'$ is used for the standard value of the average $\bar{\bar{X}}$. The standard value $\bar{X}'$ is usually equal to the value of $\bar{\bar{X}}$ at the time that the process was brought under satisfactory control. Other standard values listed in the Glossary of Z1.3 are σ', p', $p'n$, and c'.

The method used in Z1.3 for establishing standard values is to determine a value σ', of the standard deviation σ for individual items, by means of R and $\bar{\sigma}$ and an appropriate table value. The two formulas for computing σ' are

$$\sigma' = \frac{\bar{\sigma}}{c_2} \tag{1}$$

$$\sigma' = \frac{\bar{R}}{d_2} \tag{2}$$

where c_2 and d_2 are table values that depend upon the sample size. These may be found in Table III. If the process is under control, the control will be satisfactory if the value of σ', from either (1) or (2), is such that $\bar{\bar{X}} + 3\sigma'$ is less than the upper specification limit, and $\bar{\bar{X}} - 3\sigma'$ is greater than the lower specification limit. If these conditions are true then $\bar{\bar{X}}$ is adopted as the standard value for the average and is denoted by $\bar{X}'$. Hence the values obtained from $\bar{X}' \pm 3\sigma'$ must fall inside of specification limits.[1]

The control limits in terms of standard values are

$$\bar{X}' \pm A\sigma' \tag{3}$$

Values of A for various sample sizes are given in Table III.

With $\bar{X}' = \bar{\bar{X}}$ and $\sigma' = \bar{R}/d_2$, formula (3) gives the same location of control limits as found by

$$\bar{\bar{X}} \pm A_2R \tag{4}$$

In Art. 23, the averages $\bar{\bar{X}}$ and $\bar{R}$ which corresponded to a satisfactory condition of control were adopted as standard values and limits determined from (4) were projected forward to be used for the control of quality during future production.

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[1] The relationship between $\bar{X}' \pm 3\sigma'$ and specification limits, as expressed in this paragraph, is that given in the texts that were referred to above. From Art. 22g it may be seen that this relationship is true only if end values given in specifications are unacceptable values.

The formulas

$$\overline{\overline{X}} \pm \sqrt{n}Y_A = \overline{\overline{X}} \pm \sqrt{n}A_2\bar{R} \tag{5}$$

by which danger limits were located in Art. 22, correspond to the formulas in (3), because

$$\sqrt{n}A_2\bar{R} = 3\sigma' \tag{6}$$

Hence danger limits that were used in Chap. III are in reality $3\sigma'$ limits for individual units.

In the case of charts for fraction defective, for number of defectives, and for number of defects, the same considerations for the determination of standard values were given in Chap. IV of the present text as are given in Z1.3. The chief difference is that in the present text prime values have not been indicated.

46. Texts

Any one who understands and can apply the formulas and the methods in Chaps. III and IV should be able to make or interpret control charts. If he wished further study he should be ready to read texts that cover the subject of statistical quality control more completely than has been done in this introductory presentation. A few of the more important references are listed below.

WALTER A. SHEWHART, *Economic Control of Quality of Manufactured Product*, D. Van Nostrand Company, Inc., New York, 1931.

EUGENE L. GRANT, *Statistical Quality Control*, McGraw-Hill Book Company, Inc., New York, 1946.

Guide for Quality Control and *Control Chart Method of Analyzing Data*, American War Standard Z1.1—1941 and Z1.2—1941, American Standards Association, New York.

Control Chart Method of Controlling Quality During Production, American War Standard Z1.3—1942, American Standards Association, New York.

A.S.T.M. Manual on Presentation of Data, American Society for Testing Materials, Philadelphia, Pa.

LESLIE E. SIMON, *An Engineers' Manual of Statistical Methods*, John Wiley & Sons, Inc., New York.

PAUL PEACH, *An Introduction to Industrial Statistics and Quality Control*, L. B. Phillips Company, Raleigh, N. C.

TABLES

TABLE I.—TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000 *

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
1	1	1.0000	1.000000	**51**	2601	7.1414	.019608
2	4	1.4142	.500000	**52**	2704	7.2111	.019231
3	9	1.7321	.333333	**53**	2809	7.2801	.018868
4	16	2.0000	.250000	**54**	2916	7.3485	.018519
5	25	2.2361	.200000	**55**	3025	7.4162	.018182
6	36	2.4495	.166667	**56**	3136	7.4833	.017857
7	49	2.6458	.142857	**57**	3249	7.5498	.017544
8	64	2.8284	.125000	**58**	3364	7.6158	.017241
9	81	3.0000	.111111	**59**	3481	7.6811	.016949
10	100	3.1623	.100000	**60**	3600	7.7460	.016667
11	121	3 3166	.090909	**61**	3721	7.8102	.016393
12	144	3.4641	.083333	**62**	3844	7.8740	.016129
13	169	3.6056	.076923	**63**	3969	7.9373	.015873
14	196	3 7417	.071429	**64**	4096	8.0000	.015625
15	225	3.8730	.066667	**65**	4225	8.0623	.015385
16	256	4.0000	.062500	**66**	4356	8.1240	.015152
17	289	4 1231	.058824	**67**	4489	8.1854	.014925
18	324	4 2426	.055556	**68**	4624	8.2462	.014706
19	361	4 3589	.052632	**69**	4761	8.3066	.014493
20	400	4 4721	.050000	**70**	4900	8.3666	.014286
21	441	4.5826	.047619	**71**	5041	8.4261	.014085
22	484	4 6904	.045455	**72**	5184	8.4853	.013889
23	529	4 7958	.043478	**73**	5329	8.5440	.013699
24	576	4.8990	.041667	**74**	5476	8 6023	.013514
25	625	5.0000	.040000	**75**	5625	8.6603	.013333
26	676	5.0990	.038462	**76**	5776	8 7178	.013158
27	729	5.1962	.037037	**77**	5929	8.7750	.012987
28	784	5.2915	.035714	**78**	6084	8.8318	.012821
29	841	5.3852	.034483	**79**	6241	8 8882	.012658
30	900	5.4772	.033333	**80**	6400	8.9443	.012500
31	961	5.5678	.032258	**81**	6561	9.0000	.012346
32	1024	5.6569	.031250	**82**	6724	9 0554	.012195
33	1089	5.7446	.030303	**83**	6889	9.1104	.012048
34	1156	5.8310	.029412	**84**	7056	9.1652	.011905
35	1225	5.9161	.028571	**85**	7225	9.2195	.011765
36	1296	6 0000	.027778	**86**	7396	9.2736	.011628
37	1369	6.0828	.027027	**87**	7569	9.3274	.011494
38	1444	6 1644	.026316	**88**	7744	9.3808	.011364
39	1521	6 2450	.025641	**89**	7921	9.4340	.011236
40	1600	6.3246	.025000	**90**	8100	9.4868	.011111
41	1681	6 4031	.024390	**91**	8281	9.5394	.010989
42	1764	6.4807	.023810	**92**	8464	9.5917	.010870
43	1849	6.5574	.023256	**93**	8649	9.6437	.010753
44	1936	6.6332	.022727	**94**	8836	9.6954	.010638
45	2025	6.7082	.022222	**95**	9025	9.7468	.010526
46	2116	6 7823	.021739	**96**	9216	9 7980	.010417
47	2209	6 8557	.021277	**97**	9409	9 8489	.010309
48	2304	6.9282	.020833	**98**	9604	9 8995	.010204
49	2401	7 0000	.020408	**99**	9801	9 9499	.010101
50	2500	7 0711	.020000	**100**	10000	10 0000	010000

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TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
101	10201	10 0499	.009901	**151**	22801	12.2882	.006623
102	10404	10.0995	.009804	**152**	23104	12.3288	.006579
103	10609	10.1489	.009709	**153**	23409	12.3693	.006536
104	10816	10.1980	.009615	**154**	23716	12.4097	.006494
105	11025	10.2470	.009524	**155**	24025	12.4499	.006452
106	11236	10.2956	.009434	**156**	24336	12.4900	.006410
107	11449	10.3441	.009346	**157**	24649	12.5300	.006369
108	11664	10.3923	.009259	**158**	24964	12.5698	.006329
109	11881	10.4403	.009174	**159**	25281	12 6095	.006289
110	12100	10.4881	.009091	**160**	25600	12.6491	.006250
111	12321	10.5357	.009009	**161**	25921	12.6886	.006211
112	12544	10.5830	.008929	**162**	26244	12.7279	.006173
113	12769	10 6301	.008850	**163**	26569	12.7671	.006135
114	12996	10.6771	.008772	**164**	26896	12.8062	.006098
115	13225	10.7238	.008696	**165**	27225	12.8452	.006061
116	13456	10.7703	.008621	**166**	27556	12 8841	.006024
117	13689	10.8167	.008547	**167**	27889	12 9228	.005988
118	13924	10.8628	.008475	**168**	28224	12 9615	005952
119	14161	10.9087	.008403	**169**	28561	13 0000	.005917
120	14400	10.9545	.008333	**170**	28900	13.0384	.005882
121	14641	11.0000	.008264	**171**	29241	13 0767	.005848
122	14884	11.0454	.008197	**172**	29584	13 1149	.005814
123	15129	11 0905	008130	**173**	29929	13 1529	005780
124	15376	11.1355	.008065	**174**	30276	13 1909	005747
125	15625	11.1803	.008000	**175**	30625	13.2288	.005714
126	15876	11.2250	.007937	**176**	30976	13 2665	.005682
127	16129	11.2694	007874	**177**	31329	13.3041	.005650
128	16384	11.3137	.007813	**178**	31684	13 3417	.005618
129	16641	11.3578	.007752	**179**	32041	13.3791	005587
130	16900	11 4018	.007692	**180**	32400	13 4164	005556
131	17161	11 4455	.007634	**181**	32761	13 4536	.005525
132	17424	11.4891	.007576	**182**	33124	13 4907	.005495
133	17689	11.5326	.007519	**183**	33489	13.5277	005464
134	17956	11.5758	.007463	**184**	33856	13 5647	.005435
135	18225	11.6190	.007407	**185**	34225	13.6015	.005405
136	18496	11.6619	.007353	**186**	34596	13 6382	005376
137	18769	11.7047	007299	**187**	34969	13.6748	005348
138	19044	11 7473	.007246	**188**	35344	13 7113	.005319
139	19321	11 7898	007194	**189**	35721	13 7477	.005291
140	19600	11.8322	.007143	**190**	36100	13 7840	.005263
141	19881	11.8743	.007092	**191**	36481	13.8203	.005236
142	20164	11.9164	.007042	**192**	36864	13.8564	.005208
143	20449	11.9583	.006993	**193**	37249	13.8924	005181
144	20736	12.0000	.006944	**194**	37636	13 9284	.005155
145	21025	12.0416	.006897	**195**	38025	13.9642	.005128
146	21316	12.0830	.006849	**196**	38416	14 0000	005102
147	21609	12 1244	006803	**197**	38809	14 0357	.005076
148	21904	12 1655	.006757	**198**	39204	14.0712	.005051
149	22201	12 2066	.006711	**199**	39601	14 1067	.005025
150	22500	12.2474	.006667	**200**	40000	14 1421	.005000

TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
201	40401	14.1774	.004975	**251**	63001	15.8430	.003984
202	40804	14.2127	.004950	**252**	63504	15.8745	.003968
203	41209	14.2478	.004926	**253**	64009	15.9060	.003953
204	41616	14.2829	.004902	**254**	64516	15.9374	.003937
205	42025	14.3178	.004878	**255**	65025	15.9687	.003922
206	42436	14.3527	.004854	**256**	65536	16.0000	.003906
207	42849	14.3875	.004831	**257**	66049	16.0312	.003891
208	43264	14.4222	.004808	**258**	66564	16.0624	.003876
209	43681	14.4568	.004785	**259**	67081	16.0935	.003861
210	44100	14.4914	.004762	**260**	67600	16.1245	.003846
211	44521	14.5258	.004739	**261**	68121	16.1555	.003831
212	44944	14.5602	.004717	**262**	68644	16.1864	.003817
213	45369	14.5945	.004695	**263**	69169	16.2173	.003802
214	45796	14.6287	.004673	**264**	69696	16.2481	.003788
215	46225	14.6629	.004651	**265**	70225	16.2788	.003774
216	46656	14.6969	.004630	**266**	70756	16.3095	.003759
217	47089	14.7309	.004608	**267**	71289	16.3401	.003745
218	47524	14.7648	.004587	**268**	71824	16.3707	.003731
219	47961	14.7986	.004566	**269**	72361	16.4012	.003717
220	48400	14.8324	.004545	**270**	72900	16.4317	.003704
221	48841	14.8661	.004525	**271**	73441	16.4621	.003690
222	49284	14.8997	.001505	**272**	73984	16.4924	.003676
223	49729	14.9332	.004484	**273**	74529	16.5227	.003663
224	50176	14.9666	.004464	**274**	75076	16.5529	.003650
225	50625	15.0000	.004444	**275**	75625	16.5831	.003636
226	51076	15.0333	.004425	**276**	76176	16.6132	.003623
227	51529	15.0665	.004405	**277**	76729	16.6433	.003610
228	51984	15.0997	.004386	**278**	77284	16.6733	.003597
229	52441	15.1327	.004367	**279**	77841	16.7033	.003584
230	52900	15.1658	.004348	**280**	78400	16.7332	.003571
231	53361	15.1987	.004329	**281**	78961	16.7631	.003559
232	53824	15.2315	.004310	**282**	79524	16.7929	.003546
233	54289	15.2643	.004292	**283**	80089	16.8226	.003534
234	54756	15.2971	.004274	**284**	80656	16.8523	.003521
235	55225	15.3297	.004255	**285**	81225	16.8819	.003509
236	55696	15.3623	.004237	**286**	81796	16.9115	.003497
237	56169	15.3948	.004219	**287**	82369	16.9411	.003484
238	56644	15.4272	.004202	**288**	82944	16.9706	.003472
239	57121	15.4596	.004184	**289**	83521	17.0000	.003460
240	57600	15.4919	.004167	**290**	84100	17.0294	.003448
241	58081	15.5242	.004149	**291**	84681	17.0587	.003436
242	58564	15.5563	.004132	**292**	85264	17.0880	.003425
243	59049	15.5885	.004115	**293**	85849	17.1172	.003413
244	59536	15.6205	.004098	**294**	86436	17.1464	.003401
245	60025	15.6525	.004082	**295**	87025	17.1756	.003390
246	60516	15.6844	.004065	**296**	87616	17.2047	.003378
247	61009	15.7162	.004049	**297**	88209	17.2337	.003367
248	61504	15.7480	.004032	**298**	88804	17.2627	.003356
249	62001	15.7797	.004016	**299**	89401	17.2916	.003344
250	62500	15.8114	.004000	**300**	90000	17.3205	.003333

Table of Squares, Square Roots, and Reciprocals—1 to 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
301	90601	17.3494	.003322	**351**	123201	18.7350	.002849
302	91204	17.3781	.003311	**352**	123904	18.7617	.002841
303	91809	17.4069	.003300	**353**	124609	18.7883	.002833
304	92416	17.4356	.003289	**354**	125316	18.8149	.002825
305	93025	17.4642	.003279	**355**	126025	18.8414	.002817
306	93636	17.4929	.003268	**356**	126736	18.8680	.002809
307	94249	17.5214	.003257	**357**	127449	18.8944	.002801
308	94864	17.5499	.003247	**358**	128164	18.9209	.002793
309	95481	17.5784	.003236	**359**	128881	18.9473	.002786
310	96100	17.6068	.003226	**360**	129600	18.9737	.002778
311	96721	17.6352	.003215	**361**	130321	19.0000	.002770
312	97344	17.6635	.003205	**362**	131044	19.0263	.002762
313	97969	17.6918	.003195	**363**	131769	19.0526	.002755
314	98596	17.7200	.003185	**364**	132496	19.0788	.002747
315	99225	17.7482	.003175	**365**	133225	19.1050	.002740
316	99856	17.7764	.003165	**366**	133956	19.1311	.002732
317	100489	17.8045	.003155	**367**	134689	19.1572	.002725
318	101124	17.8326	.003145	**368**	135424	19.1833	.002717
319	101761	17.8606	.003135	**369**	136161	19.2094	.002710
320	102400	17.8885	.003125	**370**	136900	19.2354	.002703
321	103041	17.9165	.003115	**371**	137641	19.2614	.002695
322	103684	17.9444	.003106	**372**	138384	19.2873	.002688
323	104329	17.9722	.003096	**373**	139129	19.3132	.002681
324	104976	18.0000	.003086	**374**	139876	19.3391	.002674
325	105625	18.0278	.003077	**375**	140625	19.3649	.002667
326	106276	18.0555	.003067	**376**	141376	19.3907	.002660
327	106929	18.0831	.003058	**377**	142129	19.4165	.002653
328	107584	18.1108	.003049	**378**	142884	19.4422	.002646
329	108241	18.1384	.003040	**379**	143641	19.4679	.002639
330	108900	18.1659	.003030	**380**	144400	19.4936	.002632
331	109561	18.1934	.003021	**381**	145161	19.5192	.002625
332	110224	18.2209	.003012	**382**	145924	19.5448	.002618
333	110889	18.2483	.003003	**383**	146689	19.5704	.002611
334	111556	18.2757	.002994	**384**	147456	19.5959	.002604
335	112225	18.3030	.002985	**385**	148225	19.6214	.002597
336	112896	18.3303	.002976	**386**	148996	19.6469	.002591
337	113569	18.3576	.002967	**387**	149769	19.6723	.002584
338	114244	18.3848	.002959	**388**	150544	19.6977	.002577
339	114921	18.4120	.002950	**389**	151321	19.7231	.002571
340	115600	18.4391	.002941	**390**	152100	19.7484	.002564
341	116281	18.4662	.002933	**391**	152881	19.7737	.002558
342	116964	18.4932	.002924	**392**	153664	19.7990	.002551
343	117649	18.5203	.002915	**393**	154449	19.8242	.002545
344	118336	18.5472	.002907	**394**	155236	19.8494	.002538
345	119025	18.5742	.002899	**395**	156025	19.8746	.002532
346	119716	18.6011	.002890	**396**	156816	19.8997	.002525
347	120409	18.6279	.002882	**397**	157609	19.9249	.002519
348	121104	18.6548	.002874	**398**	158404	19.9499	.002513
349	121801	18.6815	.002865	**399**	159201	19.9750	.002506
350	122500	18.7083	.002857	**400**	160000	20.0000	.002500

Table of Squares, Square Roots, and Reciprocals—1 to 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
401	160801	20.0250	.002494	**451**	203401	21.2368	.002217
402	161604	20.0499	.002488	**452**	204304	21.2603	.002212
403	162409	20.0749	.002481	**453**	205209	21.2838	.002208
404	163216	20.0998	.002475	**454**	206116	21.3073	.002203
405	164025	20.1246	.002469	**455**	207025	21.3307	.002198
406	164836	20.1494	.002463	**456**	207936	21.3542	.002193
407	165649	20.1742	.002457	**457**	208849	21.3776	.002188
408	166464	20.1990	.002451	**458**	209764	21.4009	.002183
409	167281	20.2237	.002445	**459**	210681	21.4243	.002179
410	168100	20.2485	.002439	**460**	211600	21.4476	.002174
411	168921	20.2731	.002433	**461**	212521	21.4709	.002169
412	169744	20.2978	.002427	**462**	213444	21.4942	.002165
413	170569	20.3224	.002421	**463**	214369	21.5174	.002160
414	171396	20.3470	.002415	**464**	215296	21.5407	.002155
415	172225	20.3715	.002410	**465**	216225	21.5639	.002151
416	173056	20.3961	.002404	**466**	217156	21.5870	.002146
417	173889	20.4206	.002398	**467**	218089	21.6102	.002141
418	174724	20.4450	.002392	**468**	219024	21.6333	.002137
419	175561	20.4695	.002387	**469**	219961	21.6564	.002132
420	176400	20.4939	.002381	**470**	220000	21.6795	.002128
421	177241	20.5183	.002375	**471**	221841	21.7025	.002123
422	178084	20.5426	.002370	**472**	222784	21.7256	.002119
423	178029	20.5670	.002364	**473**	223729	21.7486	.002114
424	179776	20.5913	.002358	**474**	224676	21.7715	.002110
425	180625	20.6155	.002353	**475**	225625	21.7945	.002105
426	181476	20.6398	.002347	**476**	226576	21.8174	.002101
427	182329	20.6640	.002342	**477**	227529	21.8403	.002096
428	183184	20.6882	.002336	**478**	228484	21.8632	.002092
429	184041	20.7123	.002331	**479**	229441	21.8861	.002088
430	184900	20.7364	.002326	**480**	230400	21.9089	.002083
431	185761	20.7605	.002320	**481**	231361	21.9317	.002079
432	186624	20.7846	.002315	**482**	232324	21.9545	.002075
433	187489	20.8087	.002309	**483**	233289	21.9773	.002070
434	188356	20.8327	.002304	**484**	234256	22.0000	.002066
435	189225	20.8567	.002299	**485**	235225	22.0227	.002062
436	190096	20.8806	.002294	**486**	236196	22.0454	.002058
437	190969	20.9045	.002288	**487**	237169	22.0681	.002053
438	191844	20.9284	.002283	**488**	238144	22.0907	.002049
439	192721	20.9523	.002278	**489**	239121	22.1133	.002045
440	193600	20.9762	.002273	**490**	240100	22.1359	.002041
441	194481	21.0000	.002268	**491**	241081	22.1585	.002037
442	195364	21.0238	.002262	**492**	242064	22.1811	.002033
443	196249	21.0476	.002257	**493**	243049	22.2036	.002028
444	197136	21.0713	.002252	**494**	244036	22.2261	.002024
445	198025	21.0950	.002247	**495**	245025	22.2486	.002020
446	198916	21.1187	.002242	**496**	246016	22.2711	.002016
447	199809	21.1424	.002237	**497**	247009	22.2935	.002012
448	200704	21.1660	.002232	**498**	248004	22.3159	.002008
449	201601	21.1896	.002227	**499**	249001	22.3383	.002004
450	202500	21.2132	.002222	**500**	250000	22.3607	.002000

TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
501	251001	22.3830	.001996	**551**	303601	23.4734	.001815
502	252004	22.4054	.001992	**552**	304704	23 4947	.001812
503	253009	22.4277	001988	**553**	305809	23.5160	.001808
504	254016	22 4499	.001984	**554**	306916	23 5372	.001805
505	255025	22.4722	.001980	**555**	308025	23 5584	.001802
506	256036	22.4944	.001976	**556**	309136	23 5797	.001799
507	257049	22 5167	.001972	**557**	310249	23 6008	.001795
508	258064	22.5389	.001969	**558**	311364	23.6220	.001792
509	259081	22 5610	.001965	**559**	312481	23.6432	.001789
510	260100	22.5832	.001961	**560**	313600	23.6643	.001786
511	261121	22.6053	.001957	**561**	314721	23 6854	.001783
512	262144	22.6274	.001953	**562**	315844	23 7065	.001779
513	263169	22.6495	.001949	**563**	316969	23 7276	.001776
514	264196	22 6716	.001946	**564**	318096	23.7487	.001773
515	265225	22.6936	.001942	**565**	319225	23 7697	.001770
516	266256	22.7156	.001938	**566**	320356	23.7908	.001767
517	267289	22.7376	.001934	**567**	321489	23 8118	.001764
518	268324	22 7596	.001931	**568**	322624	23 8328	.001761
519	269361	22 7816	.001927	**569**	323761	23 8537	.001757
520	270400	22 8035	.001923	**570**	324900	23 8747	.001754
521	271441	22 8254	.001919	**571**	326041	23 8956	.001751
522	272484	22 8473	.001916	**572**	327184	23 9160	.001748
523	273529	22.8692	.001912	**573**	328329	23 9374	.001745
524	274576	22.8910	.001908	**574**	329476	23 9583	.001742
525	275625	22.9129	.001905	**575**	330625	23 9792	.001739
526	276676	22.9347	.001901	**576**	331776	24 0000	.001736
527	277729	22.9565	.001898	**577**	332929	24 0208	.001733
528	278784	22 9783	.001894	**578**	334084	24 0416	.001730
529	279841	23 0000	.001890	**579**	335241	24 0624	.001727
530	280900	23 0217	.001887	**580**	336400	24 0832	.001724
531	281961	23.0434	.001883	**581**	337561	24 1030	.001721
532	283024	23 0651	.001880	**582**	338724	24 1247	.001718
533	284089	23.0868	.001876	**583**	339889	24 1454	.001715
534	285156	23.1084	.001873	**584**	341056	24 1661	.001712
535	286225	23 1301	.001869	**585**	342225	24.1868	.001709
536	287296	23 1517	.001866	**586**	343396	24 2074	.001706
537	288369	23 1733	.001862	**587**	344569	24 2281	.001704
538	289444	23 1948	.001859	**588**	345744	24 2487	.001701
539	290521	23.2164	.001855	**589**	346921	24 2693	.001698
540	291600	23 2379	.001852	**590**	348100	24 2899	.001695
541	292681	23 2594	.001848	**591**	349281	24 3105	.001692
542	293764	23 2809	.001845	**592**	350464	24 3311	.001689
543	294849	23.3024	.001842	**593**	351649	24 3516	.001686
544	295936	23.3238	.001838	**594**	352836	24.3721	.001684
545	297025	23 3452	.001835	**595**	354025	24.3926	.001681
546	298116	23 3666	.001832	**596**	355216	24 4131	.001678
547	299209	23.3880	.001828	**597**	356409	24.4336	.001675
548	300304	23.4094	001825	**598**	357604	24 4540	.001672
549	301401	23 4307	.001821	**599**	358801	24 4745	.001669
550	302500	23.4521	.001818	**600**	360000	24.4949	.001667

TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
601	361201	24.5153	.001664	**651**	423801	25.5147	.001536
602	362404	24.5357	.001661	**652**	425104	25.5343	.001534
603	363609	24.5561	.001658	**653**	426409	25.5539	.001531
604	364816	24.5764	.001656	**654**	427716	25.5734	.001529
605	366025	24.5967	.001653	**655**	429025	25.5930	.001527
606	367236	24.6171	.001650	**656**	430336	25.6125	.001524
607	368449	24.6374	.001647	**657**	431649	25.6320	.001522
608	369664	24.6577	.001645	**658**	432964	25.6515	.001520
609	370881	24.6779	.001642	**659**	434281	25.6710	.001517
610	372100	24.6982	.001639	**660**	435600	25.6905	.001515
611	373321	24.7184	.001637	**661**	436921	25.7099	.001513
612	374544	24.7386	.001634	**662**	438244	25.7294	.001511
613	375769	24.7588	.001631	**663**	439569	25.7488	.001508
614	376996	24.7790	.001629	**664**	440896	25.7682	.001506
615	378225	24.7992	.001626	**665**	442225	25.7876	.001504
616	379456	24.8193	.001623	**666**	443556	25.8070	.001502
617	380689	24.8395	.001621	**667**	444889	25.8263	.001499
618	381924	24.8596	.001618	**668**	446224	25.8457	.001497
619	383161	24.8797	.001616	**669**	447561	25.8650	.001495
620	384400	24.8998	.001613	**670**	448900	25.8844	.001493
621	385641	24.9199	.001610	**671**	450241	25.9037	.001490
622	386884	24.9399	.001608	**672**	451584	25.9230	.001488
623	388129	24.9600	.001605	**673**	452929	25.9422	.001486
624	389376	24.9800	.001603	**674**	454276	25.9615	.001484
625	390625	25.0000	.001600	**675**	455625	25.9808	.001481
626	391876	25.0200	.001597	**676**	456976	26.0000	.001479
627	393129	25.0400	.001595	**677**	458329	26.0192	.001477
628	394384	25.0599	.001592	**678**	459684	26.0384	.001475
629	395641	25.0799	.001590	**679**	461041	26.0576	.001473
630	396900	25.0998	.001587	**680**	462400	26.0768	.001471
631	398161	25.1197	.001585	**681**	463761	26.0960	.001468
632	399424	25.1396	.001582	**682**	465124	26.1151	.001466
633	400689	25.1595	.001580	**683**	466489	26.1343	.001464
634	401956	25.1794	.001577	**684**	467856	26.1534	.001462
635	403225	25.1992	.001575	**685**	469225	26.1725	.001460
636	404496	25.2190	.001572	**686**	470596	26.1916	.001458
637	405769	25.2389	.001570	**687**	471969	26.2107	.001456
638	407044	25.2587	.001567	**688**	473344	26.2298	.001453
639	408321	25.2784	.001565	**689**	474721	26.2488	.001451
640	409600	25.2982	.001563	**690**	476100	26.2679	.001449
641	410881	25.3180	.001560	**691**	477481	26.2869	.001447
642	412164	25.3377	.001558	**692**	478864	26.3059	.001445
643	413449	25.3574	.001555	**693**	480249	26.3249	.001443
644	414736	25.3772	.001553	**694**	481636	26.3439	.001441
645	416025	25.3969	.001550	**695**	483025	26.3629	.001439
646	417316	25.4165	.001548	**696**	484416	26.3818	.001437
647	418609	25.4362	.001546	**697**	485809	26.4008	.001435
648	419904	25.4558	.001543	**698**	487204	26.4197	.001433
649	421201	25.4755	.001541	**699**	488601	26.4386	.001431
650	422500	25.4951	.001538	**700**	490000	26.4575	.001429

Table of Squares, Square Roots, and Reciprocals—1 to 1000.—(*Continued*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
701	491401	26.4764	.001427	**751**	564001	27.4044	.001332
702	492804	26.4953	.001425	**752**	565504	27.4226	.001330
703	494209	26 5141	.001422	**753**	567009	27.4408	.001328
704	495616	26.5330	.001420	**754**	568516	27.4591	.001326
705	497025	26.5518	.001418	**755**	570025	27.4773	.001325
706	498436	26 5707	.001416	**756**	571536	27 4955	.001323
707	499849	26 5895	.001414	**757**	573049	27.5136	.001321
708	501264	26 6083	.001412	**758**	574564	27 5318	.001319
709	502681	26 6271	.001410	**759**	576081	27.5500	.001318
710	504100	26.6458	.001408	**760**	577600	27.5681	.001316
711	505521	26 6646	.001406	**761**	579121	27.5862	.001314
712	506944	26 6833	.001404	**762**	580644	27 6043	.001312
713	508369	26 7021	001403	**763**	582169	27 6225	.001311
714	509796	26 7208	.001401	**764**	583696	27 6405	.001309
715	511225	26 7395	.001399	**765**	585225	27.6586	.001307
716	512656	26.7582	.001397	**766**	586756	27.6767	.001305
717	514089	26 7769	.001395	**767**	588289	27.6948	.001304
718	515524	26 7955	.001393	**768**	589824	27.7128	.001302
719	516961	26.8142	.001391	**769**	591361	27.7308	.001300
720	518400	26.8328	.001389	**770**	592900	27.7489	.001299
721	519841	26 8514	.001387	**771**	594441	27 7669	.001297
722	521284	26 8701	.001385	**772**	595984	27 7849	.001295
723	522729	26.8887	.001383	**773**	597529	27 8029	.001294
724	524176	26.9072	.001381	**774**	599076	27.8209	.001292
725	525625	26 9258	.001379	**775**	600625	27.8388	.001290
726	527076	26.9444	.001377	**776**	602176	27.8568	.001289
727	528529	26 9629	.001376	**777**	603729	27 8747	.001287
728	529984	26 9815	.001374	**778**	605284	27 8927	001285
729	531441	27 0000	.001372	**779**	606841	27 9106	001284
730	532900	27.0185	.001370	**780**	608400	27.9285	.001282
731	534361	27.0370	.001368	**781**	609961	27 9464	.001280
732	535824	27 0555	.001366	**782**	611524	27 9643	.001279
733	537289	27 0740	.001364	**783**	613089	27 9821	.001277
734	538756	27.0024	.001362	**784**	614656	28.0000	.001276
735	540225	27.1109	.001361	**785**	616225	28 0179	.001274
736	541696	27.1293	.001359	**786**	617796	28.0357	.001272
737	543169	27 1477	.001357	**787**	619369	28.0535	.001271
738	544644	27 1662	.001355	**788**	620944	28 0713	.001269
739	546121	27 1846	.001353	**789**	622521	28.0891	.001267
740	547600	27.2029	.001351	**790**	624100	28.1069	.001266
741	549081	27.2213	.001350	**791**	625681	28.1247	.001264
742	550564	27 2397	.001348	**792**	627264	28 1425	.001263
743	552049	27.2580	.001346	**793**	628849	28 1603	.001261
744	553536	27.2764	.001344	**794**	630436	28.1780	.001259
745	555025	27.2947	.001342	**795**	632025	28.1957	.001258
746	556516	27 3130	.001340	**796**	633616	28 2135	001256
747	558009	27.3313	.001339	**797**	635209	28.2312	.001255
748	559504	27 3496	.001337	**798**	636804	28 2489	.001253
749	561001	27.3679	.001335	**799**	638401	28.2666	.001252
750	562500	27.3861	.001333	**800**	640000	28 2843	.001250

TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000.—(*Continued*)

Numbers	Squares	Square roots	Recipro-cals	Numbers	Squares	Square roots	Recipro-cals
801	641601	28 3019	.001248	**851**	724201	29 1719	001175
802	643204	28 3196	.001247	**852**	725904	29 1890	.001174
803	644809	28 3373	001245	**853**	727609	29.2062	.001172
804	646416	28.3549	001244	**854**	729316	29 2233	.001171
805	648025	28.3725	.001242	**855**	731025	29 2404	.001170
806	649636	28 3901	.001241	**856**	732736	29 2575	.001168
807	651249	28 4077	.001239	**857**	734449	29 2746	001167
808	652864	28 4253	001238	**858**	736164	29 2916	.001166
809	654481	28 4429	001236	**859**	737881	29 3087	.001164
810	656100	28.4605	.001235	**860**	739600	29.3258	.001163
811	657721	28 4781	.001233	**861**	741321	29 3428	.001161
812	659344	28.4956	001232	**862**	743044	29 3598	.001160
813	660969	28.5132	001230	**863**	744769	29.3769	001159
814	662596	28 5307	.001229	**864**	746496	29 3939	.001157
815	664225	28.5482	001227	**865**	748225	29.4109	.001156
816	665856	28 5657	.001225	**866**	749956	29.4279	.001155
817	667489	28 5832	001224	**867**	751689	29 4449	.001153
818	669124	28 6007	001222	**868**	753424	29.4618	001152
819	670761	28 6182	001221	**869**	755161	29 4788	001151
820	672400	28 6356	.001220	**870**	756900	29 4958	.001149
821	674041	28 6531	001218	**871**	758641	29.5127	.001148
822	675684	28 6705	001217	**872**	760384	29.5296	.001147
823	677329	28 6880	001215	**873**	762129	29 5466	001145
824	678976	28 7054	001214	**874**	763876	29 5635	001144
825	680625	28 7228	.001212	**875**	765625	29 5804	.001143
826	682276	28 7402	001211	**876**	767376	29 5973	001142
827	683929	28 7576	001209	**877**	769129	29 6142	001140
828	685584	28 7750	001208	**878**	770884	29 6311	001139
829	687241	28 7924	.001206	**879**	772641	29 6479	.001138
830	688900	28.8097	.001205	**880**	774400	29.6648	.001136
831	690561	28 8271	001203	**881**	776161	29 6816	.001135
832	692224	28 8444	001202	**882**	777924	29 6985	.001134
833	693889	28 8617	001200	**883**	779689	29 7153	001133
834	695556	28 8791	001199	**884**	781456	29.7321	.001131
835	697225	28 8964	001198	**885**	783225	29 7489	.001130
836	698896	28 9137	001196	**886**	784996	29 7658	.001129
837	700569	28 9310	.001195	**887**	786769	29.7825	001127
838	702244	28 9482	001193	**888**	788544	29 7993	.001126
839	703921	28 9655	.001192	**889**	790321	29 8161	.001125
840	705600	28 9828	001190	**890**	792100	29 8329	001124
841	707281	29 0000	001189	**891**	793881	29 8496	.001122
842	708964	29 0172	.001188	**892**	795664	29 8664	.001121
843	710649	29 0345	.001186	**893**	797449	29 8831	001120
844	712336	29 0517	001185	**894**	799236	29 8998	.001119
845	714025	29 0689	.001183	**895**	801025	29.9166	.001117
846	715716	29.0861	.001182	**896**	802816	29 9323	001116
847	717409	29.1033	.001181	**897**	804609	29 9500	001115
848	719104	29 1204	.001179	**898**	806404	29 9666	001114
849	720801	29 1376	.001178	**899**	808201	29 9833	001112
850	722500	29 1548	.001176	**900**	810000	30.0000	001111

TABLE OF SQUARES, SQUARE ROOTS, AND RECIPROCALS—1 TO 1000.—(*Concluded*)

Numbers	Squares	Square roots	Reciprocals	Numbers	Squares	Square roots	Reciprocals
901	811801	30.0167	.001110	**951**	904401	30.8383	.001052
902	813604	30.0333	.001109	**952**	906304	30.8545	.001050
903	815409	30.0500	.001107	**953**	908209	30.8707	.001049
904	817216	30.0666	.001106	**954**	910116	30.8869	.001048
905	819025	30.0832	.001105	**955**	912025	30.9031	.001047
906	820836	30.0998	.001104	**956**	913936	30.9192	.001046
907	822649	30.1164	.001103	**957**	915849	30.9354	.001045
908	824464	30.1330	.001101	**958**	917764	30.9516	.001044
909	826281	30.1496	.001100	**959**	919681	30.9677	.001043
910	828100	30.1662	.001099	**960**	921600	30.9839	.001042
911	829921	30.1828	.001098	**961**	923521	31.0000	.001041
912	831744	30.1993	.001096	**962**	925444	31.0161	.001040
913	833569	30.2159	.001095	**963**	927369	31.0322	.001038
914	835396	30.2324	.001094	**964**	929296	31.0483	.001037
915	837225	30.2490	.001093	**965**	931225	31.0644	.001036
916	839056	30.2655	.001092	**966**	933156	31.0805	.001035
917	840889	30.2820	.001091	**967**	935089	31.0966	.001034
918	842724	30.2985	.001089	**968**	937024	31.1127	.001033
919	844561	30.3150	.001088	**969**	938961	31.1288	.001032
920	846400	30.3315	.001087	**970**	940900	31.1448	.001031
921	848241	30.3480	.001086	**971**	942841	31.1609	.001030
922	850084	30.3645	.001085	**972**	944784	31.1769	.001029
923	851929	30.3809	.001083	**973**	946729	31.1929	.001028
924	853776	30.3974	.001082	**974**	948676	31.2090	.001027
925	855625	30.4138	.001081	**975**	950625	31.2250	.001026
926	857476	30.4302	.001080	**976**	952576	31.2410	.001025
927	859329	30.4467	.001079	**977**	954529	31.2570	.001024
928	861184	30.4631	.001078	**978**	956484	31.2730	.001022
929	863041	30.4795	.001076	**979**	958441	31.2890	.001021
930	864900	30.4959	.001075	**980**	960400	31.3050	.001020
931	866761	30.5123	.001074	**981**	962361	31.3209	.001019
932	868624	30.5287	.001073	**982**	964324	31.3369	.001018
933	870489	30.5450	.001072	**983**	966289	31.3528	.001017
934	872356	30.5614	.001071	**984**	968256	31.3688	.001016
935	874225	30.5778	.001070	**985**	970225	31.3847	.001015
936	876096	30.5941	.001068	**986**	972196	31.4006	.001014
937	877969	30.6105	.001067	**987**	974169	31.4166	.001013
938	879844	30.6268	.001066	**988**	976144	31.4325	.001012
939	881721	30.6431	.001065	**989**	978121	31.4484	.001011
940	883600	30.6594	.001064	**990**	980100	31.4643	.001010
941	885481	30.6757	.001063	**991**	982081	31.4802	.001009
942	887364	30.6920	.001062	**992**	984064	31.4960	.001008
943	889249	30.7083	.001060	**993**	986049	31.5119	.001007
944	891136	30.7246	.001059	**994**	988036	31.5278	.001006
945	893025	30.7409	.001058	**995**	990025	31.5436	.001005
946	894916	30.7571	.001057	**996**	992016	31.5595	.001004
947	896809	30.7734	.001056	**997**	994009	31.5753	.001003
948	898704	30.7896	.001055	**998**	996004	31.5911	.001002
949	900601	30.8058	.001054	**999**	998001	31.6070	.001001
950	902500	30.8221	.001053	**1000**	1000000	31.6228	.001000

TABLE II.—SPREAD VALUES FOR CONTROL LIMITS CHART FOR FRACTION DEFECTIVE

$$Y_p = 3\sqrt{\frac{\bar{p}(1-\bar{p})}{\bar{n}}}$$

$\bar{n}$ \ $\bar{p}$	0.005	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
1	0.2115	0.2985	0.4200	0.5118	0.5879	0.6538	0.7125	0.7654	0.8139	0.8585	0.9000
2	0.1492	0.2111	0.2963	0.3619	0.4157	0.4623	0.5038	0.5412	0.5755	0.6071	0.6364
3	0.1220	0.1723	0.2425	0.2955	0.3394	0.3775	0.4113	0.4419	0.4690	0.4957	0.5196
4	0.1058	0.1493	0.2100	0.2559	0.2939	0.3269	0.3562	0.3827	0.4069	0.4293	0.4500
5	0.0945	0.1335	0.1878	0.2289	0.2629	0.2924	0.3186	0.3423	0.3640	0.3839	0.4025
6	0.0864	0.1216	0.1715	0.2089	0.2400	0.2669	0.2909	0.3125	0.3323	0.3505	0.3674
7	0.0800	0.1128	0.1587	0.1934	0.2222	0.2471	0.2693	0.2888	0.3076	0.3245	0.3402
8	0.0749	0.1055	0.1484	0.1809	0.2078	0.2312	0.2519	0.2706	0.2878	0.3035	0.3182
9	0.0705	0.0995	0.1400	0.1706	0.1960	0.2179	0.2375	0.2551	0.2713	0.2862	0.3000
10	0.0670	0.0944	0.1328	0.1618	0.1859	0.2068	0.2253	0.2421	0.2573	0.2715	0.2846
12	0.0611	0.0862	0.1212	0.1477	0.1697	0.1887	0.2057	0.2210	0.2349	0.2478	0.2598
14	0.0566	0.0798	0.1122	0.1367	0.1571	0.1745	0.1904	0.2046	0.2175	0.2295	0.2405
16	0.0529	0.0746	0.1050	0.1279	0.1470	0.1635	0.1781	0.1914	0.2035	0.2146	0.2250
18	0.0499	0.0703	0.0990	0.1206	0.1386	0.1541	0.1679	0.1804	0.1918	0.2024	0.2121
20	0.0473	0.0667	0.0939	0.1144	0.1315	0.1462	0.1593	0.1712	0.1820	0.1920	0.2012
22	0.0451	0.0636	0.0896	0.1091	0.1253	0.1394	0.1519	0.1632	0.1735	0.1830	0.1919
25	0.0423	0.0597	0.0840	0.1024	0.1176	0.1308	0.1425	0.1531	0.1628	0.1717	0.1800
30	0.0386	0.0545	0.0767	0.0934	0.1073	0.1195	0.1301	0.1398	0.1486	0.1567	0.1643
35	0.0357	0.0504	0.0710	0.0865	0.0994	0.1105	0.1204	0.1294	0.1374	0.1453	0.1521
40	0.0334	0.0472	0.0664	0.0809	0.0930	0.1034	0.1126	0.1210	0.1287	0.1357	0.1423
50	0.0299	0.0422	0.0594	0.0724	0.0831	0.0925	0.1007	0.1082	0.1151	0.1214	0.1273
60	0.0273	0.0385	0.0542	0.0661	0.0759	0.0844	0.0920	0.0988	0.1051	0.1108	0.1162
70	0.0253	0.0357	0.0502	0.0612	0.0703	0.0781	0.0852	0.0915	0.0973	0.1026	0.1076
80	0.0237	0.0334	0.0470	0.0572	0.0657	0.0731	0.0796	0.0856	0.0910	0.0960	0.1006
90	0.0223	0.0315	0.0443	0.0539	0.0620	0.0689	0.0751	0.0807	0.0858	0.0905	0.0949
100	0.0212	0.0298	0.0420	0.0512	0.0588	0.0654	0.0712	0.0765	0.0814	0.0859	0.0900
125	0.0189	0.0267	0.0376	0.0458	0.0526	0.0585	0.0637	0.0685	0.0728	0.0768	0.0805
150	0.0172	0.0244	0.0342	0.0418	0.0480	0.0534	0.0582	0.0625	0.0665	0.0701	0.0735
200	0.0149	0.0211	0.0297	0.0362	0.0416	0.0462	0.0504	0.0541	0.0576	0.0607	0.0636
250	0.0134	0.0188	0.0266	0.0324	0.0372	0.0414	0.0451	0.0484	0.0515	0.0543	0.0569
300	0.0122	0.0172	0.0242	0.0294	0.0339	0.0377	0.0411	0.0442	0.0470	0.0496	0.0520
400	0.0106	0.0149	0.0210	0.0256	0.0294	0.0327	0.0356	0.0383	0.0407	0.0429	0.0450
600	0.0086	0.0122	0.0172	0.0208	0.0240	0.0267	0.0291	0.0313	0.0332	0.0350	0.0367
800	0.0075	0.0105	0.0149	0.0181	0.0208	0.0231	0.0252	0.0271	0.0288	0.0304	0.0318
1,000	0.0067	0.0094	0.0133	0.0161	0.0186	0.0207	0.0225	0.0242	0.0257	0.0271	0.0285
1,200	0.0061	0.0086	0.0121	0.0147	0.0170	0.0189	0.0206	0.0221	0.0235	0.0248	0.0260
1,500	0.0055	0.0077	0.0109	0.0132	0.0152	0.0169	0.0184	0.0198	0.0210	0.0222	0.0232
2,000	0.0047	0.0067	0.0094	0.0114	0.0131	0.0146	0.0159	0.0171	0.0182	0.0192	0.0201
2,500	0.0042	0.0060	0.0084	0.0102	0.0118	0.0131	0.0142	0.0153	0.0163	0.0172	0.0180
3,000	0.0039	0.0054	0.0077	0.0095	0.0107	0.0119	0.0130	0.0140	0.0149	0.0157	0.0164
4,000	0.0033	0.0047	0.0066	0.0081	0.0093	0.0103	0.0113	0.0121	0.0129	0.0136	0.0142
6,000	0.0027	0.0038	0.0054	0.0066	0.0076	0.0084	0.0092	0.0099	0.0105	0.0111	0.0116
8,000	0.0024	0.0033	0.0047	0.0057	0.0066	0.0073	0.0080	0.0086	0.0091	0.0096	0.0101
10,000	0.0021	0.0030	0.0042	0.0051	0.0059	0.0065	0.0071	0.0077	0.0081	0.0086	0.0090
12,000	0.0019	0.0027	0.0038	0.0047	0.0054	0.0060	0.0065	0.0070	0.0074	0.0079	0.0082
15,000	0.0017	0.0024	0.0034	0.0042	0.0048	0.0053	0.0058	0.0062	0.0066	0.0070	0.0073
20,000	0.0015	0.0021	0.0030	0.0036	0.0042	0.0046	0.0050	0.0054	0.0058	0.0061	0.0064

Table entries have been given to four decimal places, but they may be used to two, three, or four places, as desired.

For an explanation of the use of the table see Art. 37.

TABLE III.—FACTORS FOR COMPUTING CONTROL LIMITS

Number of observations in sample, n	Chart for averages			Chart for standard deviations					Chart for ranges					
	Factors for control limits			Factor for central line	Factors for control limits				Factor for central line	Factors for control limits				
	A	A_1	A_2	c_2	B_1	B_2	B_3	B_4	d_2	D	D_1	D_2	D_3	D_4
2	2.121	3.759	1.880	0.5642	0	2.064	0	3.658	1.128	2.268	0	3.686	0	3.268
3	1.732	2.394	1.023	0.7236	0	1.948	0	2.692	1.693	1.574	0	4.358	0	2.574
4	1.500	1.880	0.729	0.7979	0	1.859	0	2.330	2.059	1.282	0	4.698	0	2.282
5	1.342	1.596	0.577	0.8407	0	1.789	0	2.128	2.326	1.114	0	4.918	0	2.114
6	1.225	1.410	0.483	0.8686	0.003	1.735	0.003	1.997	2.534	1.004	0	5.078	0	2.004
7	1.134	1.277	0.419	0.8882	0.086	1.690	0.097	1.903	2.704	0.924	0.205	5.203	0.076	1.924
8	1.061	1.175	0.373	0.9027	0.153	1.653	0.169	1.831	2.847	0.864	0.387	5.307	0.136	1.864
9	1.000	1.094	0.337	0.9139	0.207	1.621	0.227	1.774	2.970	0.816	0.546	5.394	0.184	1.816
10	0.949	1.028	0.308	0.9227	0.252	1.594	0.273	1.727	3.078	0.777	0.687	5.469	0.223	1.777

The above table is a copy of entries 2 to 10 of Table I on p. 50 of Supplement B to the A.S.T.M. Manual on Presentation of Data (which is cited in Art. 46), courtesy of the American Society for Testing Materials. Column D has been added.

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